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On the Schwarzschild Solution: a Review

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Abstract**—***In this paper the well-known Schwarzschild Solution is discussed. In the first section, by resorting, as usual, to the Einstein Field Equations, a short summary of the conventional derivation is provided. In the second section, we carry out an alternative derivation of the Schwarzschild Metric. The above-mentioned procedure is based upon several noteworthy hypotheses, among which the existence of a further spatial dimension stands out. Initially, we postulate a Universe identifiable with a 4-ball, homogeneously filled with matter, whose radius equates the Schwarzschild Radius. Then, in order to obtain the vacuum field, all the available mass is ideally concentrated in a single point. By imposing a specific condition concerning the measured radius, we deduce a metric that, if subjected to an appropriate parametrization, allows us to finally obtain the Schwarzschild solution.*

Keywords**—*****Vacuum Field, Weak Field Approximation, Schwarzschild Metric, Alternative Derivation.***

I. CONVENTIONAL DERIVATION

If we impose a spherical symmetry, the general static solution is represented by the underlying metric:

$$ds^2 = A(r)c^2dt^2 - B(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\varphi^2 \quad (1)$$

Obviously, we have already set equal to one, without any loss of generality, the parametric coefficient related to the angular part of the metric. A and B exclusively depend on the "flat coordinate", denoted by r. In particular, coherently with the hypothesized static scenario, whatever time derivatives must necessarily vanish.

As for the metric tensor, from *(1)* we immediately obtain:

$$g_{ij} = \begin{bmatrix} A(r) & 0 & 0 & 0 \\ 0 & -B(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix} \quad (2)$$

$$g^{ij} = \begin{bmatrix} \frac{1}{A(r)} & 0 & 0 & 0 \\ 0 & -\frac{1}{B(r)} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2\theta} \end{bmatrix} \quad (3)$$

Let's deduce the *Christoffel Symbols*. Generally, we have:

$$\Gamma^k_{ij} = \frac{1}{2}g^{kh}\left(\frac{\partial g_{hi}}{\partial x^j} + \frac{\partial g_{hj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^h}\right) \quad (4)$$

The indexes run from *0* to *3*. Clearly, *0* stands for t, *1* for r, *2* for θ, and *3* for φ.

Setting $k=0$, from *(2)*, *(3)* and *(4)*, we obtain:

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2A}\frac{dA}{dr} \quad (5)$$

All the other symbols (if $k=0$) vanish.

Setting $k=1$, from *(2)*, *(3)* and *(4)*, we obtain:

$$\Gamma^1_{00} = \frac{1}{2B}\frac{dA}{dr},\ \Gamma^1_{11} = \frac{1}{2B}\frac{dB}{dr},\ \Gamma^1_{12} = -\frac{r}{B},\ \Gamma^1_{13} = -\frac{r}{B}\sin^2\theta \quad (6)$$

All the other symbols (if $k=1$) vanish.

Setting $k=2$, from *(2)*, *(3)* and *(4)*, we obtain:

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r},\ \Gamma^2_{33} = -\sin\theta\cos\theta \quad (7)$$

All the other symbols (if $k=2$) vanish.

Setting $k=3$, from *(2)*, *(3)* and *(4)*, we obtain:

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r},\ \Gamma^3_{23} = \Gamma^3_{23} = \frac{1}{\tan\theta} \quad (8)$$

All the other symbols (if $k=3$) vanish.

Let's now deduce the components of the *Ricci Tensor*. Generally, with obvious meaning of the notation, we have:

$$R_{ij} = \frac{\partial \Gamma^k_{ik}}{\partial x^j} - \frac{\partial \Gamma^k_{ij}}{\partial x^k} + \Gamma^l_{ik}\Gamma^k_{jl} - \Gamma^l_{ij}\Gamma^k_{kl} \quad (9)$$

By means of some simple mathematical passages, omitted for brevity, we obtain all the non-vanishing components:

$$R_{00} = -\frac{1}{2B}\frac{d^2A}{dr^2} + \frac{1}{4B}\frac{dA}{dr}\left(\frac{1}{A}\frac{dA}{dr} + \frac{1}{B}\frac{dB}{dr}\right) - \frac{1}{rB}\frac{dA}{dr} \quad (10)$$

$$R_{11} = \frac{1}{2A}\frac{d^2A}{dr^2} - \frac{1}{4A}\frac{dA}{dr}\left(\frac{1}{A}\frac{dA}{dr} + \frac{1}{B}\frac{dB}{dr}\right) - \frac{1}{rB}\frac{dB}{dr} \quad (11)$$

$$R_{22} = \frac{1}{B} + \frac{r}{2B}\left(\frac{1}{A}\frac{dA}{dr} - \frac{1}{B}\frac{dB}{dr}\right) - 1 \quad (12)$$

$$R_{33} = \sin^2\theta\left[\frac{1}{B} + \frac{r}{2B}\left(\frac{1}{A}\frac{dA}{dr} - \frac{1}{B}\frac{dB}{dr}\right) - 1\right] = \sin^2\theta\, R_{22} \quad (13)$$

If we denote with R the *Ricci Scalar* and with T_{ij} the generic component of the *Stress-Energy Tensor*, the *Einstein Field Equations* [1] can be written as follows:

$$R_{ij} - \frac{1}{2} R g_{ij} = \frac{8\pi G}{c^4} T_{ij} \tag{14}$$

If we impose that, outside the mass that produces the field, there is the "absolute nothing" (neither matter nor energy), the first member of *(14)*, that represents the so-called *Einstein Tensor*, must vanish. Consequently, we have:

$$R_{ij} - \frac{1}{2} R g_{ij} = 0 \tag{15}$$

From *(15)*, exploiting the fact that the Einstein Tensor and the Ricci Tensor are trace-reverse of each other, we have:

$$R_{ij} = 0 \tag{16}$$

From *(10)*, *(11)* and *(16)*, we immediately obtain:

$$-\frac{1}{2AB}\frac{d^2A}{dr^2} + \frac{1}{4AB}\frac{dA}{dr}\left(\frac{1}{A}\frac{dA}{dr} + \frac{1}{B}\frac{dB}{dr}\right) - \frac{1}{rAB}\frac{dA}{dr} = 0 \tag{17}$$

$$\frac{1}{2AB}\frac{d^2A}{dr^2} - \frac{1}{4AB}\frac{dA}{dr}\left(\frac{1}{A}\frac{dA}{dr} + \frac{1}{B}\frac{dB}{dr}\right) - \frac{1}{rB^2}\frac{dB}{dr} = 0 \tag{18}$$

From *(17)* and *(18)*, we have:

$$\frac{dB}{B} = -\frac{dA}{A} \tag{19}$$

$$B = \frac{K_1}{A} \tag{20}$$

The value of the constant K_1 can be deduced by imposing that, at infinity, the ordinary flat metric must be recovered. In other terms, we must impose the following condition:

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1 \tag{21}$$

From *(20)*, taking into account *(21)*, we obtain:

$$B = \frac{1}{A} \tag{22}$$

$$g_{00} g_{11} = -1 \tag{23}$$

From *(16)* and *(12)* we have:

$$A + \frac{rA}{2}\left[\frac{1}{A}\frac{dA}{dr} - A\frac{d}{dr}\left(\frac{1}{A}\right)\right] - 1 = 0 \tag{24}$$

$$A + r\frac{dA}{dr} - 1 = \frac{d}{dr}(rA) - 1 = 0 \tag{25}$$

$$A = 1 + \frac{K_2}{r} \tag{26}$$

It is worth highlighting that, at this point, we could already deduce the "original" *Schwarzschild Solution* [2], without assigning any particular value to the constant K_2.

The value of K_2 can be directly deduced by resorting to the so-called *Weak Field Approximation*. If we denote with ϕ the *Gravitational Potential*, we can write:

$$A = g_{00} = \left(1 - \frac{\phi}{c^2}\right)^2 \cong 1 - 2\frac{\phi}{c^2} = 1 - \frac{2MG}{rc^2} \tag{27}$$

From *(26)* and *(27)*, we immediately deduce:

$$K_2 = -\frac{2MG}{c^2} \tag{28}$$

From *(22)* and *(27)*, we have:

$$B = \frac{1}{1 - \frac{2MG}{rc^2}} \tag{29}$$

At this point, the metric can be immediately written. However, in order to directly obtain a more compact form, we can denote with R_s the value of r that makes the metric singular (the so-called *Schwarzschild radius*):

$$\frac{2MG}{c^2} = R_s \tag{30}$$

From *(27)*, *(29)* and *(30)* we finally obtain:

$$ds^2 = \left(1 - \frac{R_s}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - \frac{R_s}{r}} - r^2 (d\theta^2 - \sin^2\theta \, d\varphi^2) \tag{31}$$

II. ALTERNATIVE DERIVATION

Although the space we are allowed to perceive is curved, since it is identifiable with a hyper-sphere whose radius depends on our state of motion [3] [4], the Universe in its entirety, assimilated to a 4-ball, is considered as being flat [5]. All the points are replaced by straight line segments: in other terms, what we perceive as being a point is actually a straight-line segment crossing the centre of the 4-ball. [4] At the beginning (no singularity), we hypothesize that the Universe, whose radius equates the Schwarzschild Radius, is homogeneously filled with matter [6].

The scenario is qualitatively depicted in *Figure 4*.

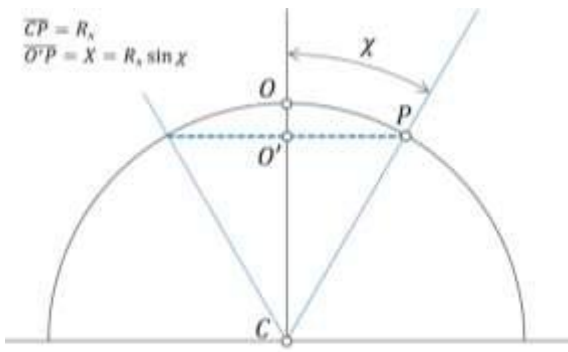

Figure 1. Initial Scenario

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If we denote with X the predicted radius (the straight-line segment bordered by O' and P in *Figure 1*), and with χ the angular distance between P and O (as perceived by an ideal observer placed in C), we can write:

$$X = R_s \sin\chi \tag{32}$$

The measure of the corresponding great circumference, denoted by C_X, is provided by the following banal relation:

$$C_X = 2\pi X = 2\pi R_s \sin\chi \tag{33}$$

From *(32)* we immediately deduce:

$$\chi = \arcsin\left(\frac{X}{R_s}\right) \tag{34}$$

If we denote with l the measured radius (the arc bordered by O and P in *Figure 1*), from *(34)* we have:

$$dl = R_s d\chi = \frac{dX}{\sqrt{1-\left(\frac{X}{R_s}\right)^2}} \tag{35}$$

At this point, the *Friedmann–Robertson–Walker* metric can be finally written:

$$ds^2 = c^2dt^2 - \frac{dX^2}{1-\left(\frac{X}{R_s}\right)^2} - X^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{36}$$

Let's now suppose that all the available mass may be concentrated in the origin. According to our model [6], the spatial lattice undergoes a deformation that drags O to C.

The scenario is qualitatively depicted in *Figure 2*.

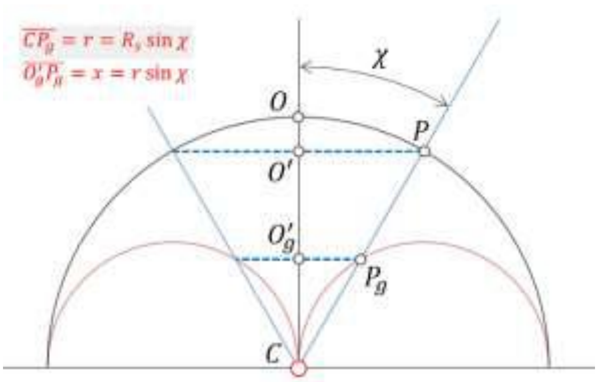

Figure 2. Singularity

We hypothesize that the singularity does not influence the measured distance (the proper radius): in other terms, if the angular distance between whatever couple of points does not undergo any variation, the corresponding measured distance remains the same [6]. In order to satisfy this condition, the radial coordinate (the segment bordered by C and P_g in *Figure 2*) must abide by the following relation:

$$r = R_s \sin\chi \tag{37}$$

In fact, it can be instantly verified that:

$$\sqrt{\left(\frac{dr}{d\chi}\right)^2 + r^2} = R_s \tag{38}$$

The predicted radius (the segment bordered by O'_g and P_g in *Figure 2*), denoted by x, undergoes a reduction. Taking into account *(37)*, we can immediately write:

$$x = r\sin\chi = R_s \sin^2\chi = X\sin\chi = \frac{X^2}{R_s} \tag{39}$$

The measure of the corresponding great circumference, denoted by C_x, is provided by the following banal relation:

$$C_x = 2\pi x = 2\pi X \sin\chi \tag{40}$$

We can now write the following metric:

$$ds^2 = c^2dt^2 - \frac{dX^2}{1-\left(\frac{X}{R_s}\right)^2} - \frac{X^4}{R_s^2}(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{41}$$

Exploiting *(39)*, the angular distance can be evidently expressed as follows:

$$\chi = \arcsin\left(\sqrt{\frac{x}{R_s}}\right) \tag{42}$$

Consequently, we have:

$$dl = R_s d\chi = R_s d\left[\arcsin\left(\sqrt{\frac{x}{R_s}}\right)\right] = R_s \frac{d\left(\sqrt{\frac{x}{R_s}}\right)}{\sqrt{1-\left(\sqrt{\frac{x}{R_s}}\right)^2}} \tag{43}$$

$$dl = \frac{1}{2}\frac{\sqrt{\frac{R_s}{x}}\,dx}{\sqrt{1-\frac{x}{R_s}}} \tag{44}$$

Taking into account *(44)*, we can write the metric in *(41)* as a function of x:

$$ds^2 = c^2dt^2 - \frac{1}{4}\frac{\frac{R_s}{x}dx^2}{1-\frac{x}{R_s}} - x^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{45}$$

According to our model [6], once again, the proper radius remains the same, notwithstanding the singularity: on the contrary, the predicted radius, as well as the corresponding great circumference, undergoes a contraction [6] [7].

Let's now suppose that we want to "warp" (not parameterize) the previous metric. If we impose that the measure of the predicted radius must remain the same (if

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we assign a greater value to the "quantum of space"), we have to consequently increase the value of the proper radius. Obviously, in order to keep the speed of light constant, we are forced to assign a smaller value to the "quantum of time" (in particular, the more we approach the singularity, the more time must flow slowly) [7] [8]. Taking into account the fact that the relation between the initial predicted radius and the reduced one is expressed by *(39)*, we obtain the following metric, in which time, very evidently, is no longer considered as being absolute:

$$ds^2 = \left(\frac{X}{R_s}\right)^2 c^2 dt^2 - \left(\frac{R_s}{X}\right)^2 \frac{dX^2}{1-\left(\frac{X}{R_s}\right)^2} - X^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \quad (46)$$

Obviously, the metric we have just obtained is far different from the Schwarzschild solution. Suffice it to notice that, very evidently, both X and x cannot tend to infinity.

In order to recover the Schwarzschild solution, we have to carry out a parameterization [6]. On this purpose, we need to find two new coordinates, denoted by $R^*(\chi)$ and $r^*(\chi)$, that may satisfy the following relations:

$$r^* = R^* \sin\chi \quad (47)$$

$$dl_{r^*} = \sqrt{r^{*2} + \left(\frac{dr^*}{d\chi}\right)^2}\, d\chi = \sqrt{R^{*2} + \left(\frac{dR^*}{d\chi}\right)^2}\, d\chi = dl_{R^*} \quad (48)$$

From *(47)* we have:

$$\frac{dr^*}{d\chi} = \frac{dR^*}{d\chi}\sin\chi + R^*\cos\chi \quad (49)$$

$$r^{*2} + \left(\frac{dr^*}{d\chi}\right)^2 = R^{*2} + \left(\frac{dR^*}{d\chi}\right)^2 \sin^2\chi + 2R^*\frac{dR^*}{d\chi}\sin\chi\cos\chi \quad (50)$$

Very evidently, if the derivative of R^* is null, imposing $R^*(0)=R_s$, we instantly recover *(37)*. On the contrary, if the derivative of R^* is not null, from *(50)* we easily deduce:

$$2\tan\chi\, d\chi = \frac{dR^*}{R^*} \quad (51)$$

Imposing $R^*(0)=R_s$, from *(51)* we immediately obtain:

$$R^* = \frac{R_s}{\cos^2\chi} \quad (52)$$

From *(52)*, we have:

$$\sin\chi = \sqrt{1 - \frac{R_s}{R^*}} \quad (53)$$

$$dR^* = 2R_s \frac{\sin\chi}{\cos^3\chi} d\chi \quad (54)$$

From *(48)*, *(52)* and *(54)*, denoting with l^* the parameterized proper radius (we must bear in mind that, according to the model herein proposed, the singularity does not influence the measured distance), we obtain:

$$dl_{R^*} = dl_{r^*} = 2R_s \frac{\sin\chi}{\cos^3\chi}\sqrt{1 + \frac{1}{4\tan^2\chi}}\, d\chi = \sqrt{1 + \frac{1}{4\tan^2\chi}}\, dR^* = dl^* \quad (55)$$

From *(54)* and *(55)* we have:

$$\lim_{\chi\to\pi/2} \frac{dl^*}{dR^*} = 1 \quad (56)$$

Taking into account *(47)* and *(52)*, the parameterized predicted radius is provided by the following relation:

$$X^* = R^*\sin\chi = R_s \frac{\sin\chi}{\cos^2\chi} = r^* \quad (57)$$

As for the corresponding great circumference, we have:

$$C_{X^*} = 2\pi X^* = 2\pi r^* \quad (58)$$

From *(54)* and *(57)* we have:

$$\lim_{\chi\to\pi/2} \frac{X^*}{R^*} = 1 \quad (59)$$

Taking into account *(47)* and *(52)*, the parameterized reduced predicted radius can be written as follows:

$$x^* = r^*\sin\chi = R_s\tan^2\chi = R^* - R_s \quad (60)$$

As for the corresponding great circumference, we have:

$$C_{x^*} = 2\pi x^* = 2\pi(R^* - R_s) \quad (61)$$

In *Figure 3* a useful comparison between old and new (parameterized) coordinates is qualitatively displayed.

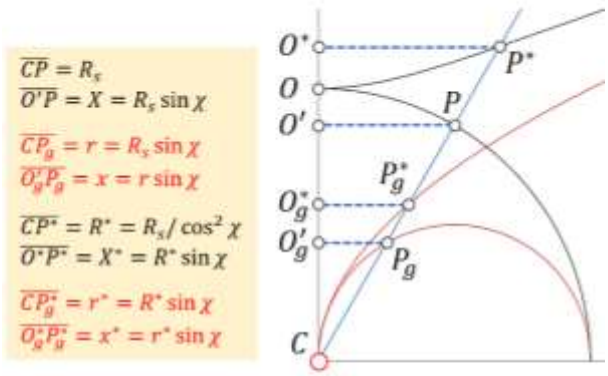

Figure 3. Singularity: Comparison Between Coordinates

In *Figure 4*, the very interesting scenario we obtain if $\chi=\pi/4$ is qualitatively depicted.

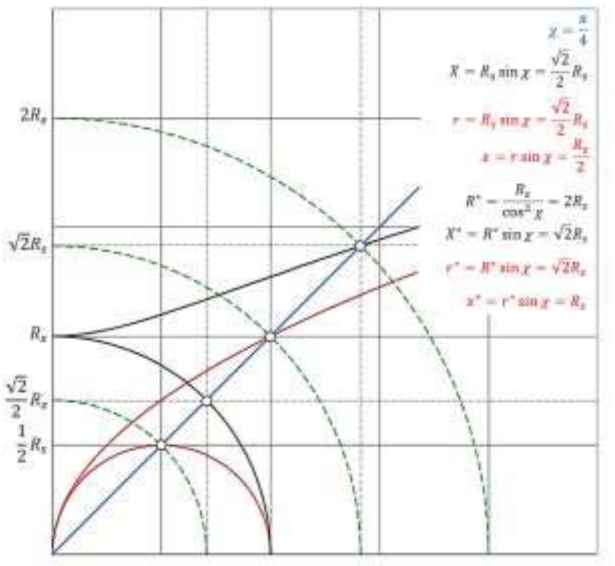

Figure 4. Singularity: particular case ($\chi=\pi/4$)

By virtue of *(55)* and *(57)*, when mass is evenly spread on the hypersphere with which we identify the Universe we are allowed to perceive (actually, when matter homogeneously fills the 4-ball with which we identify the Universe in its entirety), the parameterized distance in *(36)* acquires the underlying compact form:

$$ds^{*2}=c^2dt^{*2}-dl^{*2}-X^{*2}(d\theta^2+\sin^2\theta d\varphi^2) \qquad (62)$$

Obviously, we have replaced *t* with *t**. Time, in fact, shows necessarily trace of the parameterization we have carried out. On this subject, if we consider a null geodesic (a light-like interval) in the equatorial plane, from *(62)* we have:

$$cdt^*=dl^* \qquad (63)$$

The previous banal relation clearly shows that, if space is subjected to a parameterization, being *c* a constant, time must be parameterized too.

Far from the origin (when χ tends to $\pi/2$), taking into account *(56)* and *(59)*, the metric in *(62)* (no singularity) acquires the following "flat" form:

$$ds^{*2}=c^2dt^{*2}-dR^{*2}-R^{*2}(d\theta^2+\sin^2\theta d\varphi^2) \qquad (64)$$

Let's now concentrate in the origin all the available mass. According to our model, once again, the proper radius remains the same, notwithstanding the singularity: on the contrary, the predicted radius, as well as the corresponding great circumference, undergoes a contraction. However, following the same line of reasoning we have exploited in order to deduce *(46)*, imposing that the measure of the predicted radius must remain the same (if we assign a greater value to the "quantum of space"), we have to consequently increase the value of the proper radius. Consequently, in order to keep the speed of light constant, we are forced to assign a smaller value to the "quantum of time" (in particular, the more we approach the singularity, the more time must flow slowly). Since the relation between the initial predicted radius and the reduced one, both parameterized, is expressed by *(47)*, taking into account *(53)*, by "warping" *(65)* we obtain the following metric, that represents a Schwarzschild-like Solution:

$$ds^{*2}=\left(1-\frac{R_s}{R^*}\right)c^2dt^{*2}-\frac{dR^{*2}}{1-\frac{R_s}{R^*}} - R^{*2}(d\theta^2+\sin^2\theta d\varphi^2) \qquad (65)$$

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