Intuitionistic Fuzzy Hv-subgroups
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Abstract—In this paper we introduce the concept of intuitionistic fuzzy Hv-subgroup and prove some related results. We also consider the fundamental relation $β^*$ defined on Hv-group $H$ and for an intuitionistic fuzzy subset $A = \{μ_A, λ_A\}$ of $H$, we define an intuitionistic fuzzy subset $A^β_*$ of $H^β_*$ and we prove a theorem concerning the fundamental relation $β^*$.

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I. INTRODUCTION
The concept of hyperstructure was introduced in 1934 by Marty [10]. Hyperstructures have many applications to several branches of pure and applied sciences. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Vougiouklis [26] introduced a new class of hyperstructures, the so-called Hv-structures, in which equality is replaced by non-empty intersection. After the introduction of fuzzy sets by Zadeh [18], there have been a number of generalizations came in existence of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [17] is one of them. At the beginning according to Atanassov the term intuitionistic means that the sum of the degree of membership and the degree of non-membership is less than 1 but after some time this condition is extended and supposed to satisfy the constraint that the sum of the degree of membership and the degree of non-membership is less than or equal to 1. Basically the algebraic structure of intuitionistic fuzzy set is one of the interval-valued set not the intuitionistic logic [31]. In the same time the idea came in existence that the interval-valued sets are mathematically redundant up-to which level so by discussion [32] it is mathematically redundant up-to every level like power set level, fiber level, and categorical level. This naturally leads to interval-valued sets in a first step of departure away from standard fuzzy set. Indeed this is used in a long tradition in the field of Economics, Engineering and Science, etc. where the intervals were used to represent values of quantities in case of uncertainty. Currently it is studied in various domains of Information Technology, including preference modeling, learning and reasoning [33-34].


Here it is very important to note that Intuitionistic fuzzy set is to be combined with the study of hyperstructures for more generalization of the generalized concept. In this paper, we generalize the concept of fuzzy Hv-groups [3] by using the notion of intuitionistic fuzzy set and prove some results in this respect. We also consider the fundamental relation $β^*$ defined on Hv-group $H$ and for an intuitionistic fuzzy subset $A = \{μ_A, λ_A\}$ of $H$, we define an intuitionistic fuzzy subset $A^β_*$ of $H^β_*$ and prove a fundamental theorem concerning the group $H^β_*$.

Throughout this paper left reproduction axiom for the hypergroups is verified.

II. BASIC DEFINITIONS
In this section first we give some basic definitions for proving the further results.

**Definition 2.1**[7] Let $X$ be a non-empty set. A mapping $\mu : X \to [0, 1]$ is called a fuzzy set in $X$.

**Definition 2.2**[7] An intuitionistic fuzzy set $A$ in a non-empty set $X$ is an object having the form $A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and $\lambda_A : X \to [0, 1]$ denote the degree of membership and degree of non-membership of each element $x \in X$ to the set $A$ respectively and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$. We shall use the symbol $A = \{\mu_A, \lambda_A\}$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}$.

**Definition 2.3**[27] Let $H$ be a non–empty set and $*: H \times H \to \wp^v(H)$ be a hyperoperation, where $\wp^v(H)$ is the set of all the non-empty subsets of $H$.

The $*$ is called weak associative if $(x*y)*z \cap x*(y*z) \neq \phi$, $\forall x, y, z \in H$.

Where $A*B = \bigcup_{a \in A, b \in B} a*b$, $\forall A, B \subseteq H$.

The $*$ is called weak commutative if $x*y \cap y*x \neq \phi$, $\forall x, y \in H$.

$(H, *)$ is called an $H$–group if

(i) $*$ is weak associative.
(ii) $a*H = H*a = H$, $\forall a \in H$ (Reproduction axiom).

**Definition 2.4**[3] Let $H$ be a hypergroup (or $H$-group) and let $\mu$ be a fuzzy subset of $H$. Then $\mu$ is said to be a fuzzy subhypergroup (or fuzzy $H$-subgroup) of $H$ if the following axioms hold:

(i) $\min\{\mu(x), \mu(y)\} \leq \inf_{x \in A \cup B} \{\mu(a)\}$, $\forall x, y \in H$.

(ii) For all $x, a \in H$ there exists $y \in H$ such that $x \in a*y$ and $\min\{\mu(a), \mu(x)\} \leq \{\mu(y)\}$.

### III. INTUITIONISTIC FUZZY HV-SUBGROUP

In this section we define intuitionistic fuzzy $H$-subgroup of a hypergroup and then we obtain the relation between an intuitionistic fuzzy subhypergroup and level subhypergroup. This relation is expressed in terms of a necessary and sufficient condition.

**Definition 3.1** Let $H$ be a hypergroup (or $H$-group). An intuitionistic fuzzy set $A = \{\mu_A, \lambda_A\}$ of $H$ is called intuitionistic fuzzy subhypergroup (or intuitionistic fuzzy $H$-subgroup) of $H$ if the following axioms hold:

(i) $\min\{\mu(x), \mu(y)\} \leq \inf_{x \in A \cup B} \{\mu(a)\}$, $\forall x, y \in H$.

(ii) For all $x, a \in H$ there exists $y \in H$ such that $x \in a*y$ and $\min\{\mu(a), \mu(x)\} \leq \{\mu(y)\}$.

(iii) $\sup_{x \in A \cup B} \{\lambda(a)\} = \sup_{x \in A \cup B} \{\lambda(x)\}$.

(iv) $\forall x, a \in H$, $x = (a \cdot a^{-1}) \cdot x = a \cdot (a^{-1} \cdot x)$, hence $x \in a*(a^{-1} \cdot x)$.

Therefore it is enough to put $y = a^{-1} \cdot x$ and in this case $\lambda(x) = \lambda(a^{-1} \cdot x) \leq \max\{\lambda(x), \lambda(x)\}$.
Now suppose that $H$ is a set and $A = \{\mu_A, \lambda_A\}$ is an intuitionistic fuzzy subset of $H$. We define the hyperoperation $*: H \times H \to \phi^*(H)$ as follows:

Let $x, y \in H$ if $\mu_A(x) \leq \mu_A(y)$, then $y \ast x = x \ast y = \{t : t \in H, \mu_A(x) \leq \mu_A(t) \leq \mu_A(y)\}$.

We get

$(x \ast y) \ast (x \ast y) = x \ast (y \ast x)$

and

$(x \ast y) \ast (x \ast y) = (x \ast y) \ast (y \ast x)$

and

$(x \ast y) \ast (x \ast y) = (x \ast y) \ast (y \ast x)$

Then the set $A(x \ast y)$, such that $x \ast y \in H$, implying $x \ast y \in A(x) \ast A(y)$. We will prove the following proposition.

**Proposition 3.3** Let $(H, \cdot)$ be a group and $A = \{\mu_A, \lambda_A\}$ be an intuitionistic fuzzy subgroup of $H$, then $A = \{\mu_A, \lambda_A\}$ is an intuitionistic fuzzy $H$-subgroup of $H$.

**Proof.** In order to prove the proposition, it is sufficient to prove the conditions (i) and (ii) of definition 3.1.

(iii) Since $\forall x, y \in H$ we have

$$\sup_{a \in H} \{\lambda_a(x) = \lambda_a(y)\} = \sup_{a \in H} \{\lambda_a(x) = \max_{a \leq a} \{\lambda_a(x), \lambda_a(y)\}\}.$$

(iv) Now suppose $x, a \in H$, if $\lambda_a(x) \geq \lambda_a(y)$, then $\lambda_a(x) \geq \lambda_a(y)$.

Therefore for every $x, y \in H$ we have

$$x \in a \ast y \quad \text{and} \quad \min_{a \leq a} \{\mu_a(x), \mu_a(y)\} \leq \mu_a(y).$$

From $x \in A$ and $a \in A$, we get $\min_{a \leq a} \{\mu_a(x), \mu_a(y)\} \geq t$ and $t \ast y \in A(y)$. Then we have $A(x) \ast A(y) \subseteq A(z)$. And this proves $A(x) \ast A(y) \subseteq A(z)$.

Conversely, assume that $\forall t, 0 \leq t \leq 1$, $A_t \neq \emptyset$ is an $H$-subgroup of $H$.

(i) Now let $x, y \in H$, we can write $\mu_a(x) \geq \min_{a \leq a} \{\mu_a(x), \mu_a(y)\}$ and $\mu_a(y) \geq \min_{a \leq a} \{\mu_a(x), \mu_a(y)\}$. Then we put $t_0 = \min_{a \leq a} \{\mu_a(x), \mu_a(y)\}$, then $x \in A_{t_0}$ and $y \in A_{t_0}$, so $x \ast y \in A_{t_0}$. Therefore for every $x, y \in H$ we have $\mu_a(x) \geq t_0$ implying $\inf_{a \in H} \{\mu_a(x), \mu_a(y)\} = \min_{a \leq a} \{\mu_a(x), \mu_a(y)\}$.

(ii) Now suppose $x, a \in H$, if $\lambda_a(x) \geq \lambda_a(y)$, then $\lambda_a(x) \geq \lambda_a(y)$.

Therefore for every $x, y \in H$ we have $\lambda_a(x) \geq \lambda_a(y)$. Hence for every $a \in A$, we have $a \ast A \subseteq A$. Now let $x \in A$, then there exists $y \in H$ such that $x \in a \ast y$ and $\min_{a \leq a} \{\mu_a(x), \mu_a(y)\} \leq \mu_a(y)$. From $x \in A$ and $a \in A$, we get $\min_{a \leq a} \{\mu_a(x), \mu_a(y)\} \geq t$ and $\mu_a(y) \geq t$. And this proves $A(x) \ast A(y) \subseteq A(z)$. We can obtain the following corollary from Theorem 3.4.

**Corollary 3.5** Let $(H, \cdot)$ be an $H$-group and $A = \{\mu_A, \lambda_A\}$ be an intuitionistic fuzzy $H$-subgroup of $H$. Then

$$\forall x, y \in H, x \ast y = \inf_{a \leq a} \{\mu_a(x), \mu_a(y)\} \leq \mu_a(y).$$
If \( 0 \leq t_1 \leq t_2 \leq 1 \), then \( \mu_1 = \mu_2 \) if and only if there is no \( x \) in \( H \) such that \( t_1 \leq \mu(x) \leq t_2 \).

**Corollary 3.6** Let \((H, \cdot)\) be an \( H \)-group and \( A = \{\mu_1, \lambda_1\} \) be an intuitionistic fuzzy \( H \)-subgroup of \( H \). If the range of \( A = \{\mu_1, \lambda_1\} \) is the finite set \( \{t_1, t_2, \ldots, t_n\} \), then the set \( \{A_{t_i} : 1 \leq i \leq n\} \) contains all the level \( H \)-subgroups of \( A = \{\mu_1, \lambda_1\} \). Moreover if \( t_1 > t_2 > \ldots > t_n \), then all the level \( H \)-subgroups \( A_{t_i} \) form the following chain \( A_{t_1} \subseteq A_{t_2} \subseteq \ldots \subseteq A_{t_n} \).

**Theorem 3.7** Let \((H, \cdot)\) be an \( H \)-group. Then every \( H \)-subgroup of \( H \) is a level \( H \)-subgroup of an intuitionistic fuzzy \( H \)-subgroup of \( H \).

**Proof.** Let \( A \) be an \( H \)-subgroup of \( H \). For a fixed real number \( c, 0 < c \leq 1 \), the intuitionistic fuzzy subset \( A \) is defined as follows:

\[
A(x) = \begin{cases} 
\{c, x \in A \} & \text{if } x \in A \\
\{0, x \notin A\} & \text{if } x \notin A
\end{cases}
\]

We have \( A = A_{t_0} \) and by theorem 3.4, it is adequate to prove that \( A \) is an intuitionistic fuzzy \( H \)-subgroup. This is simple and we leave out for readers.

**Corollary 3.8** Let \((H, \cdot)\) be an \( H \)-group and \( A \) be a non-empty subset of \( H \). Then a necessary and sufficient condition for \( A \) to be an \( H \)-subgroup is that \( A = A_{t_0} \), where \( A \) is an intuitionistic fuzzy \( H \)-subgroup and \( 0 < t_0 \leq 1 \).

**Proof.** This is obvious from Theorems 3.4 and 3.7.

**Definition 3.9** Let \((H, \cdot)\) be an \( H \)-group and \( A \) be an intuitionistic fuzzy \( H \)-subgroup of \( H \). \( A \) is called right fuzzy closed with respect to \( H \) if \( \forall a, b \in H \) all the \( y \) in \( b \in a \cdot x \) satisfy \( \min \{A(b), A(a)\} \leq A(x) \). We call \( A \) left fuzzy closed with respect to \( H \) if \( \forall a, b \in H \) all the \( y \) in \( b \in y \cdot a \) satisfy \( \min \{A(b), A(a)\} \leq A(y) \).

If \( A \) is left and right fuzzy closed, then \( A \) is called fuzzy closed.

**Theorem 3.10** If the intuitionistic fuzzy \( H \)-subgroup \( A = \{\mu_1, \lambda_1\} \) is right fuzzy closed, then \( A \cdot (H - A) = H - A \).

**Proof.** If \( b \in A \cdot (H - A) \), then there exists \( a \in A \) and \( x \in H - A \) such that \( b = a \cdot x \). Therefore \( A(x) < t \leq A(a) \) and since \( A \) is right fuzzy closed we get \( \min \{A(a), A(b)\} \leq A(x) \). Hence \( A(b) \leq A(x) < t \) which implies \( b \in H - A \). So we have proved \( A \cdot (H - A) \subseteq H - A \).

On the other hand if \( x \in H - A \), then for every \( a \in A \), by the reproduction axiom there exists \( y \in H \) such that \( x \in a \cdot y \) and so it is enough to prove \( y \in H - A \). Since \( A \) is an intuitionistic fuzzy \( H \)-subgroup of \( H \), by definition we have \( \min \{A(a), A(y)\} \leq \inf \{A(a)\} \) which implies \( \min \{A(a), A(y)\} \leq A(x) \) since \( A \) is right fuzzy closed so (ii) \( \min \{A(a), A(y)\} \leq A(y) \). Now from \( x \in H - A \) we get \( a \in A \) and so \( A(x) < t \leq A(a) \).

Using (ii) we obtain \( A(x) \leq A(y) \). Therefore \( A(x) \leq \min \{A(a), A(y)\} \) and by (i) the relation \( \min \{A(a), A(y)\} = A(x) \) is obtained. But \( A(x) < A(a) \) and hence \( \min \{A(a), A(y)\} = A(y) \).

So \( A(x) = A(y) \) and since \( x \in H - A \) we get \( y \in H - A \) and the theorem is proved.

**IV. THE FUNDAMENTAL RELATION**

In this section we will prove a theorem concerning the fundamental relation \( \beta^* \). Let \((H, \cdot)\) be an \( H \)-group. The relation \( \beta^* \) is the smallest equivalence relation on \( H \) such that the quotient \( H/\beta^* \) is a group. \( \beta^* \) is called the fundamental equivalence relation on \( H \). This relation is studied by Corsini [22] concerning hypergroups, see also [9, 24, 27]. According to [27] if \( U \) denotes the set of all the
finite products of elements of \( H \), then a relation \( \beta \) can be defined on \( H \) whose transitive closure is the fundamental relation \( \beta^* \). The relation \( \beta \) is as follows: for \( x \) and \( y \) in \( H \) we write \( x \beta y \) if and only if \( \{ x, y \} \subseteq u \), for some \( u \in U \).

Suppose \( \beta^* \) (a) is the equivalence class containing \( a \in H \).

Then the product \( \otimes \) on \( H/\beta^* \), the set of all the equivalence classes, is defined as follows:

\[
\beta^*(a) \otimes \beta^*(b) = \{ \beta^*(c) : c \in \beta^*(a) \cdot \beta^*(b) \}, \quad \forall a, b \in H.
\]

It is proved in [27] that \( \beta^*(a) \otimes \beta^*(b) = \beta^*(c) \) for all \( c \in \beta^*(a) \cdot \beta^*(b) \). In this way \( H/\beta^* \) becomes a hypergroup. If we put \( \beta^*(a) \otimes \beta^*(b) = \beta^*(c) \), then \( H/\beta^* \) becomes a group.

**Definition 4.1** Let \( (H, \cdot) \) be an \( H \)-group and \( A = \{ \mu_A, \lambda_A \} \) be an intuitionistic fuzzy subset of \( H \). The intuitionistic fuzzy subset \( A_{\beta^*} \) on \( H/\beta^* \) is defined as follows:

\[
\begin{align*}
\mu_{\beta^*} : H/\beta^* &\to [0, 1] \\
\lambda_{\beta^*} : H/\beta^* &\to [0, 1] \\
\mu_{\beta^*} (\beta^*(x)) &= \sup_{a \in \beta^*(x)} \{ \mu(a) \} \\
\lambda_{\beta^*} (\beta^*(x)) &= \inf_{a \in \beta^*(x)} \{ \lambda(a) \}
\end{align*}
\]

The concept of T-norm has been studied in [8], and definition of T-fuzzy subgroups of a group \( G \) has been introduced in [11]. Now we define T-intuitionistic fuzzy \( H \)-subgroup as follows:

**Definition 4.2** Let \( (H, \cdot) \) be an \( H \)-group and let \( A = \{ \mu_A, \lambda_A \} \) be an intuitionistic fuzzy subset of \( H \). Then \( A \) is said to be a T-intuitionistic fuzzy \( H \)-subgroup of \( H \) with respect to T-norm \( T \) if the following axioms hold:

(i) \( T(\mu_A(x), \mu_A(y)) \leq \inf_{a \in x \cdot y} \{ \mu_A(a) \}, \quad \forall x, y \in H. \)

(ii) \( \forall x, a \in H \) there exists \( y \in H \) such that \( x \cdot a \cdot y \) and \( T(\mu_A(a), \mu_A(x)) \leq \mu_A(y). \)

(iii) \( T(\lambda_A(x), \lambda_A(y)) \geq \sup_{a \in x \cdot y} \{ \lambda_A(a) \}, \quad \forall x, y \in H. \)

(iv) \( \forall x, a \in H \) there exists \( y \in H \) such that \( x \cdot a \cdot y \) and \( T(\lambda_A(a), \lambda_A(x)) \geq \lambda_A(y). \)

Now we give a more general proof of the following theorem by using the concept of T-norm.

**Theorem 4.3** Let \( T \) be a continuous T-norm and \( A = \{ \mu_A, \lambda_A \} \) be a T-intuitionistic fuzzy \( H \)-subgroup of \( H \). Considering \( H/\beta^* \) as a hypergroup, then \( A_{\beta^*} \) is a T-intuitionistic fuzzy \( H \)-subgroup of \( H/\beta^* \).

**Proof.** If \( \mu = \mu_A \) then the conditions (i) and (ii) of definition 4.2 can be easily proved by [3].

(iii) Let \( \beta^*(x) \) and \( \beta^*(y) \) be two elements of \( H/\beta^* \). We can write:

\[
\begin{align*}
T(\lambda_{\beta^*}(\beta^*(x)), \lambda_{\beta^*}(\beta^*(y))) &= T(\inf_{a \in \beta^*(x)} \{ \lambda_A(a) \}, \inf_{b \in \beta^*(y)} \{ \lambda_A(b) \}) \\
&= \inf_{a \in \beta^*(x)} \{ T(\lambda_A(a), \lambda_A(b)) \} \geq \inf_{a \in \beta^*(x)} \{ \sup_{b \in \beta^*(y)} \{ \lambda_A(a) \} \} \\
&\geq \inf_{a \in \beta^*(x)} \{ \inf_{b \in \beta^*(y)} \{ \inf_{a \in \beta^*(x)} \{ \sup_{b \in \beta^*(y)} \{ \lambda_A(a) \} \} \} \inf_{a \in \beta^*(x)} \{ \inf_{b \in \beta^*(y)} \{ \lambda_A(a) \} \} \\
&= \inf_{a \in \beta^*(x)} \{ \lambda_{\beta^*}(\beta^*(a \cdot b)) \} \\
&= \lambda_{\beta^*}(\beta^*(a \cdot b)) = \lambda_{\beta^*}(\beta^*(a) \otimes \beta^*(b))
\end{align*}
\]

(iv) Now suppose \( \beta^*(x) \) and \( \beta^*(a) \) are two arbitrary elements of \( H/\beta^* \). Since \( A = \{ \mu_A, \lambda_A \} \) is a T-intuitionistic fuzzy \( H \)-subgroup of \( H \), it follows that for all \( r \in \beta^*(a), s \in \beta^*(x) \) there exists \( y_{r,s} \in H \) such that \( r \in s \cdot y_{r,s} \) and \( T(\lambda(r), \lambda(s)) \geq \lambda(y_{r,s}) \). From \( r \in s \cdot y_{r,s} \) it follows that \( \beta^*(s) \otimes \beta^*(y_{r,s}) = \{ \beta^*(r) \} \).
which implies \( \beta' (x) \otimes \beta' (y, r) = \{ \beta' (a) \} \). Now if \( r \in \beta' (a) \) and \( s \in \beta' (x) \), then there exists \( y, r, s \in H \) such that \( \beta' (s) \otimes \beta' (y, r, s) = \{ \beta' (r) \} \) and since \( \beta' (r) = \beta' (r) \), we get \( \beta' (s) \otimes \beta' (y, r, s) = \beta' (x) \otimes \beta' (y, r, s) \) and therefore \( \beta' (y, r, s) = \beta' (y, r, s) \). So all the \( y, r, s \) satisfying \( T(\lambda (r), \lambda (s)) \geq \lambda (y, r, s) \) belong to the same equivalence class. Now we have:

\[
T(\lambda (r), \lambda (s)) = T(\inf \{ \lambda (r) \}, \inf \{ \lambda (s) \}) = \inf \{ T(\lambda (r), \lambda (s)) \} \geq \inf \{ \lambda (y, r, s) \} \geq \inf \{ \lambda (y) \} = \lambda' (y, r, s) 
\]

**Corollary 4.4** Let \( A = \{ \mu_A, \lambda_A \} \) be an intuitionistic fuzzy \( H \)-subgroup of \( H \). Considering \( H/\beta' \) as a hypergroup, then \( A/\beta' \) is an intuitionistic fuzzy \( H \)-subgroup of \( H/\beta' \).

**Proof.** This is obvious from Theorem 4.3, because minimum function is a continuous T-norm.

**Theorem 4.5** Let \( H \) be an \( H \)-group and \( A = \{ \mu_A, \lambda_A \} \) be an intuitionistic fuzzy \( H \)-subgroup of \( H \). Then \( A/\beta' \) is an intuitionistic fuzzy subgroup of \( H/\beta' \).

**Proof.** Since \( A = \{ \mu_A, \lambda_A \} \) is an intuitionistic fuzzy \( H \)-subgroup, by Corollary 4.4, the first and second conditions of Definition 3.1 are satisfied, therefore

\[
(i) \min \{ \mu_{\beta'} (x), \mu_{\beta'} (y) \} \leq \inf \{ \mu_{\beta'} (x) \beta', \forall (x), (y) \in H/\beta' \}
\]

\[
(ii) \forall \beta' (x), \beta' (a) \in H/\beta' \there exists \beta' (y) \in H/\beta' \text{ such that } \beta' (x) = \beta' (a) \otimes \beta' (y) \text{ and } \min \{ \mu_{\beta'} (x), \mu_{\beta'} (y) \} \leq \mu_{\beta'} (x) \beta', \forall (x), (y) \in H/\beta' \}
\]

\[
(iii) \lambda_{\beta'} (x, \beta', \lambda_{\beta'} (y)) = \sup \{ \lambda_{\beta'} (x) \beta', \forall x, y \in H/\beta' \}
\]

\[
(iv) \forall \beta' (x), \beta' (a) \in H/\beta' \there exists \beta' (y) \in H/\beta' \text{ such that } \beta' (x) = \beta' (a) \otimes \beta' (y) \text{ and } \max \{ \lambda_{\beta'} (x), \lambda_{\beta'} (y) \} \leq \lambda_{\beta'} (y) \}
\]

Now considering \( \beta' (x) \), \( \omega_H \) in \( H/\beta' \), by condition (ii) above there exists \( \beta' (y) \in H/\beta' \) such that

\[
\omega_H = \beta' (x) \otimes \beta' (y) \text{ and } \min \{ \mu_{\beta'} (\omega_H), \mu_{\beta'} (\beta' (x)) \} \leq \mu_{\beta'} (\beta' (y)) \}
\]

From \( \omega_H = \beta' (x) \otimes \beta' (y) \) we obtain \( \omega_H = \beta' (y) \), where \( \omega_H \) denotes the unit of the group \( H/\beta' \).

Therefore, we get (I) \( \mu_{\beta'} (\beta' (x)) \leq \mu_{\beta'} (\omega_H) \).

Now considering \( \beta' (x) \), \( \omega_H \) in \( H/\beta' \), by condition (ii) above there exists \( \beta' (y) \in H/\beta' \) such that

\[
\omega_H = \beta' (x) \otimes \beta' (y) \text{ and } \min \{ \mu_{\beta'} (\omega_H), \mu_{\beta'} (\beta' (x)) \} \leq \mu_{\beta'} (\beta' (y)) \}
\]

From \( \omega_H = \beta' (x) \otimes \beta' (y) \) we obtain \( \omega_H = \beta' (y) \), so we get (II) \( \min \{ \mu_{\beta'} (\omega_H), \mu_{\beta'} (\beta' (x)) \} \leq \mu_{\beta'} (\beta' (x)) \).

By (I) and (II) the inequality

\[
\mu_{\beta'} (\beta' (x)) \leq \mu_{\beta'} (\beta' (x)) \}
\]

is obtained.
Now for all $\beta'(x)$ in $H/\beta'$ we prove that $\lambda_{\beta'}(\beta'(x)) \leq \lambda_{\beta'}(\beta'(x)^{-1})$. Since $\beta'(x) \in H/\beta'$ by considering $\beta'(a) = \beta'(x)$ which is obtained from the second condition there exists $\beta'(y_1)$ in $H/\beta'$ such that $\beta'(x) = \beta'(x) \otimes \beta'(y_1)$ and $\max\{\lambda_{\beta'}(\beta'(x)), \lambda_{\beta'}(\beta'(x))\} \geq \lambda_{\beta'}(\beta'(y_1))$.

From $\beta'(x) = \beta'(x) \otimes \beta'(y_1)$ we obtain $\omega_y = \beta'(y_1)$, where $\omega_y$ denotes the unit of the group $H/\beta'$. Therefore, we get (III) $\lambda_{\beta'}(\beta'(x)) \geq \lambda_{\beta'}(\omega_y)$.

Now considering $\beta'(x)$, $\omega_y$ in $H/\beta'$, by condition (iv) above there exists $\beta'(y_2)$ in $H/\beta'$ such that $\omega_y = \beta'(x) \otimes \beta'(y_2)$ and $\max\{\lambda_{\beta'}(\omega_y), \lambda_{\beta'}(\beta'(x))\} \geq \lambda_{\beta'}(\beta'(y_2))$. From $\omega_y = \beta'(x) \otimes \beta'(y_2)$ we obtain $\beta'(y_2) = \beta'(x)^{-1}$, so we get

(IV) $\max\{\lambda_{\beta'}(\omega_y), \lambda_{\beta'}(\beta'(x))\} \geq \lambda_{\beta'}(\beta'(y_2))$

By (III) and (IV) the inequality $\lambda_{\beta'}(\beta'(x)) \geq \lambda_{\beta'}(\beta'(x)^{-1})$ is obtained.

REFERENCES

[20] M. Asghari-Larimi, Homomorphism of intuitionistic $(\alpha, \beta)$-fuzzy $H_v$-submodule, The