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INVERSE OF THE GENERALIZED VANDERMONDE MATRIX VIA THE FUNDAMENTAL SYSTEM OF LINEAR DIFFERENCE EQUATIONS

CLAUDEMIR ANIZ AND MUSTAPHA RACHIDI

Instituto de Matemática INMA, UFMS, Av. Costa e Silva
Cidade Universitaria, Campo Grande - MS - Brazil

ABSTRACT. In this study we display a process for inverting the generalized Vandermonde matrix, using the analytic properties of a fundamental system related to a specific linear difference equations. We establish two approaches allowing us to provide explicit formulas for the entries of the inverse of the generalized Vandermonde matrices. To enhance the effectiveness of our the approaches, significant examples and special cases are given.

Key Words: Generalized Vandermonde matrix; Inverse of the generalized Vandermonde matrix; Linear difference equations; Analytic formulas; Fibonacci fundamental system.

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1. INTRODUCTION

The usual Vandermonde systems of equations of order r is given by,

$$\sum_{i=1}^{r} \lambda_i^n x_i = v_n, \quad n = 0, 1, \ldots, r-1, \tag{1}$$

where the x_i ($1 \leq i \leq r$) are the unknown variables, the λ_i ($1 \leq i \leq r$) are distinct real (or complex) numbers and the v_n ($0 \leq n \leq r-1$) given real (or complex) numbers. Let $m_i \geq 1$ ($1 \leq i \leq s$) be s integers and λ_i ($1 \leq i \leq s$) be distinct real (or complex) numbers. For a given real (or complex) numbers v_n ($0 \leq n \leq r-1$), where $r = m_1 + \cdots + m_s$, the related generalized Vandermonde systems of equations is defined as follows,

$$\sum_{i=1}^{s} \left(\sum_{j=0}^{m_i-1} x_{i,j} n^j \right) \lambda_i^n = v_n, \quad n = 0, 1, \ldots, r-1, \tag{2}$$

where the $x_{i,j}$ ($1 \leq i \leq s$, $0 \leq j \leq m_i - 1$) are the unknown variables. The generalized Vandermonde system of equations is also known in the literature as a nonsingular usual Vandermonde system. The generalized Vandermonde systems (1)-(2) appear in several topics of mathematics such that the linear algebra, numerical analysis and polynomial

e-mails addresses: claudemir.aniz@ufms.br, mustapha.rachidi@ufms.br, mu.rachidi@gmail.com.

approximation or interpolation. They also own several important applications in various areas of applied sciences and engineering like signal processing, statistics, coding theory and control theory (see for example [8, 10, 14], and references therein). In order to solve Vandermonde systems (1)-(2), several methods have been provided in the literature (see, for instance, [2, 6, 11, 12, 16, 20]). It was shown that solving these systems is related to the inverses of their associated matrices, called also usual and generalized Vandermonde matrix, respectively. For inverting the usual Vandermonde matrix, several methods have been considered in various studies (see [2, 6, 9, 11, 17, 19], and references therein). The search for an efficient approach for computing the inverse of the generalized Vandermonde matrix, is still an attractive topic, because of its interest in various topics of mathematics and applied sciences. Especially, two recent methods have been elaborated in [2, 12], for solving the generalized system (2). The process of [2] is base on the analytic formula (or the so-called analytic Binet formula) of the linear recursive sequences $\{v_n\}_{n\geq 0}$ of constant coefficients and order r, defined as follows,

$$v_n = a_0 v_{n-1} + a_1 v_{n-2} + \cdots + a_{r-1} v_{n-r} \text{ for every } n \geq r \tag{3}$$

$$v_n = \alpha_n \quad \text{for} \quad n = 0, 1, \ldots, r-1, \tag{4}$$

where a_0, ..., a_{r-1} are the coefficients and α_0, ..., α_{r-1} are the initial data. That is, the analytic formula of sequence (3)-(4), is given by,

$$v_n = \sum_{i=1}^{s} \left(\sum_{j=0}^{m_i-1} \beta_{i,j} n^j \right) \lambda_i^n, \tag{5}$$

where the λ_i ($1 \leq i \leq s$) are the roots of the (characteristic) polynomial $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$, of multiplicities m_1, m_2,..., m_s, respectively. The scalars $\beta_{i,j}$ ($1 \leq i \leq s$, $0 \leq j \leq m_i - 1$) are obtained by solving a linear system,

$$\sum_{i=1}^{s} \left(\sum_{j=0}^{m_i-1} \beta_{i,j} n^j \right) \lambda_i^n = \alpha_n, \ 0 \leq n \leq r-1. \tag{6}$$

We show that the system of equation (6) is nothing else but a generalized Vandermonde system (2). Indeed, the unknown variables $x_{i,j}$ ($1 \leq i \leq s$, $0 \leq j \leq m_i - 1$) are identified to the scalars $\beta_{i,j}$ ($1 \leq i \leq s$, $0 \leq j \leq m_i - 1$), namely, the two systems (2) and (6) are identical. Therefore, exhibiting explicit compact analytic formulas (5) for the general term v_n, namely, explicit formulas for the scalars $\beta_{i,j}$ ($1 \leq i \leq s$, $0 \leq j \leq m_i - 1$) without solving the generalized Vandermonde system (2), will permit us to establish explicit formula for the solution of this system, or equivalently, the entries of the inverse of the associated generalized Vandermonde matrix.

This study concerns the implantation of a process for computing explicit formulas for the entries of the inverse of generalized Vandermonde matrix, through knowledge of the analytic formulation (5) of the fundamental system related to Expression (3), considered as a difference equation. To reach our goal, we exploit the recent studies of [1, 3] in order to set up two approaches which make it possible to highlight a new explicit form of the analytical formula (5) of v_n, namely, we are going to exhibit new explicit formulas for the scalars $\beta_{i,j}$ ($1 \leq i \leq s$, $0 \leq j \leq m_i - 1$), and by the way we set out the inverse of the generalized Vandermonde matrix, or equivalently we solve the the generalized Vandermonde system (2). In order to better understand our results, we will exhibit some special cases and illustrative examples. Finally, we analyze and discuss the results issued from our two approaches, by comparing them with the literature, especially with results of [2,12].

The remainder of this paper is organized as follows. Section 2 in devoted to the properties of the fundamental system of Equation (3) and its related dynamic solution. Some explicit compact analytical formulas of the dynamic solution are proposed. Section 3 concerns the process of construction of the inverse of the generalized Vandermonde matrix, with the aid of the analytical expression of the fundamental system of Equation (3). The two main results on the inverse the generalized Vandermonde matrix related to the the generalized Vandermonde system (2), are provided in Section 4. Analysis and discussion are considered in section 5. Finally, conclusion and perspectives are given in Section 6.

2. Explicit analytic solution of the dynamic solution of (3)

2.1. **Fundamental system and its dynamic solution.** Let $\mathcal{E}^{(r)}_{\mathbb{K}}(a_0,\ldots,a_{r-1})$ be the $\mathbb{K}$-vector space of finite dimension r, of solutions of Equation (3) of coefficients a_0, …, $a_{r-1} \neq 0$. Let $\{\{v_n^{(p)}\}_{n\geq 0};\ 0 \leq p \leq r-1\}$ be the family of sequences of $\mathcal{E}^{(r)}_{\mathbb{K}}(a_0,\ldots,a_{r-1})$, indexed by p ($0 \leq p \leq r-1$) defined as follows,

$$\begin{cases} v_n^{(p)} &= a_0 v_{n-1}^{(p)} + a_1 v_{n-2}^{(p)} + \cdots + a_{r-1} v_{n-r}^{(p)}, \quad \text{for } n \geq r, \\ v_n^{(p)} &= \delta_{p,n} \qquad\qquad\qquad\qquad\qquad \text{for } 0 \leq n \leq r-1. \end{cases} \tag{7}$$

Namely, this family $\{\{v_n^{(p)}\}_{n\geq 0};\ 0 \leq p \leq r-1\}$ represents r copies of sequences (3) with mutually different sets of initial values, viz. $v_n^{(p)} = \delta_{p,n}$ ($0 \leq n \leq r-1$, $0 \leq p \leq r-1$), where $\delta_{p,n}$ is the Kronecker symbol. For every $\{v_n\}_{n\geq 0}$ in $\mathcal{E}^{(r)}_{\mathbb{K}}(a_0,\ldots,a_{r-1})$ of initial data $\alpha_0,\ldots,\ \alpha_{r-1}$. Let $\{w_n\}_{n\geq 0}$ be the sequence defined by $w_n = \sum_{p=0}^{r-1} \alpha_p v_n^{(p)}$. Then, as a linear combination of the $\{v_n^{(p)}\}_{n\geq 0}$, we can show that $\{w_n\}_{n\geq 0}$ is in $\mathcal{E}^{(r)}_{\mathbb{K}}(a_0,\ldots,a_{r-1})$, moreover we have $w_k = \alpha_k$, for $0 \leq k \leq r-1$. Therefore, using Lemma 2.1, we can verify that $v_n = w_n$, for every $n \geq 0$, namely, $v_n = \sum_{p=0}^{r-1} \alpha_p v_n^{(p)}$, for every $n \geq 0$. On the other side, suppose that $\sum_{p=0}^{r-1} \alpha_p v_n^{(p)} = 0$, for every $n \geq 0$. Since $v_n^{(p)} = \delta_{p,n}$, for $0 \leq n \leq r-1$, we show that $\sum_{p=0}^{r-1} \alpha_p v_n^{(p)} = \alpha_p = 0$ for $0 \leq p \leq r-1$. Therefore, the family $\{\{v_n^{(p)}\}_{n\geq 0};\ 0 \leq p \leq r-1\}$ is a basis of the $\mathbb{K}$-vector space $\mathcal{E}^{(r)}_{\mathbb{K}}(a_0,\ldots,a_{r-1})$. The family $\{\{v_n^{(p)}\}_{n\geq 0};\ 0 \leq p \leq r-1\}$ is known in the literature as the *fundamental system* of Equation (3), or the *Fibonacci fundamental system* of Equation (3).

For reasons of utility, we first state the following elementary lemma, concerning the equality of two elements of the space $\mathcal{E}^{(r)}_{\mathbb{K}}(a_0,\ldots,a_{r-1})$.

Lemma 2.1. *Let* $\{v_n\}_{n\geq 0}$ *and* $\{w_n\}_{n\geq 0}$ *be in* $\mathcal{E}_{\mathbb{K}}^{(r)}(a_0,\ldots,a_{r-1})$. *If* $v_k = w_k$, *for* $k=0,1,...,r-1$, *then, we have* $v_n = w_n$, *for every* $n \geq 0$.

The proof of this lemma is done by simple reasoning by induction.

Among the element of the fundamental system (7), the next theorem shows the closed relation between $\{v_n^{(r-1)}\}_{n\geq 0}$ and the other elements $\{v_n^{(p)}\}_{n\geq 0}$ for $p=0,\ldots,r-2$.

Theorem 2.2. *Let* $\{\{v_n^{(p)}\}_{n\geq 0};\ 0\leq p\leq r-1\}$ *be the fundamental system of Equation (3). Then, for every* p, *with* $p=0,\ldots,r-2$, *we have,*

$$v_n^{(p)} = a_{r-p-1}v_{n-1}^{(r-1)} + a_{r-p}v_{n-2}^{(r-1)} + \cdots + a_{r-1}v_{n-p-1}^{(r-1)}, \tag{8}$$

for every $n \geq r$.

Proof. Let first recall that the sequences $\{v_n^{(p)}\}_{n\geq 0}$ are defined by (7), namely,

$$\begin{cases} v_n^{(p)} = \delta_{p,n} \text{ for } p,\ n=0,\ldots,r-1, \\ v_n^{(p)} = a_0v_{n-1}^{(p)} + a_1v_{n-2}^{(p)} + \cdots + a_{r-1}v_{n-r}^{(p)} \text{ for } n\geq r. \end{cases}$$

Let $\{w_n^{(0)}\}_{n\geq 0}$ be the sequence defined by $w_0^{(0)} = 1$ and $w_n^{(0)} = a_{r-1}v_{n-1}^{(r-1)}$. We show that $\{w_n^{(0)}\}_{n\geq 0}$ satisfies Equation (3), with initial data,

$$w_0^{(0)} = 1 \text{ and } w_n^{(0)} = a_{r-1}v_{n-1}^{(r-1)} = 0, \text{ for every } 1\leq n\leq r-1.$$

Therefore, using Lemma 2.1, we derive $w_n^{(0)} = v_n^{(0)}$, for every $n\geq 0$, namely, we have $v_n^{(0)} = a_{r-1}v_{n-1}^{(r-1)}$, for every $n\geq 1$. For $p=1$, let $\{w_n^{(1)}\}_{n\geq 0}$ be the sequence defined by

$$w_0^{(1)} = 0,\ \ w_1^{(1)} = 1 \text{ and } w_n^{(1)} = a_{r-2}v_{n-1}^{(r-1)} + a_{r-1}v_{n-2}^{(r-1)}.$$

It is clear that $w_n^{(1)} = a_{r-2}v_{n-1}^{(r-1)} + w_{n-1}^{(0)}$, thus the sequence $\{w_n^{(1)}\}_{n\geq 0}$ satisfies Equation (3), with initial data,

$$w_0^{(1)} = 0,\ \ w_1^{(1)} = 1 \text{ and } w_n^{(1)} = a_{r-2}v_{n-1}^{(r-1)} + w_{n-1}^{(0)} = 0 \text{ for } 2\leq n\leq r-1.$$

Therefore, Lemma 2.1 shows that the two sequences $\{w_n^{(1)}\}_{n\geq 0}$ and $\{v_n^{(1)}\}_{n\geq 0}$ are identical, namely, we have $v_n^{(1)} = a_{r-2}v_{n-1}^{(r-1)} + a_{r-1}v_{n-2}^{(r-1)}$. More generally, with the aid of the similar argument, for $2\leq p\leq r-2$, we consider the sequence $\{w_n^{(p)}\}_{n\geq 0}$ defined by the initial conditions $w_0^{(p)} = \cdots = w_{p-1}^{(p)} = 0$, $w_p^{(p)} = 1$, and

$$w_n^{(p)} = a_{r-p-1}v_{n-1}^{(r-1)} + a_{r-p}v_{n-2}^{(r-1)} + \cdots + a_{r-1}v_{n-p-1}^{(r-1)}.$$

We can observe that $w_n^{(p)} = a_{r-p-1}v_{n-1}^{(r-1)} + w_{n-1}^{(p-1)}$, where $w_{n-1}^{(p-1)} = a_{r-p}v_{n-2}^{(r-1)} + \cdots + a_{r-1}v_{n-p-1}^{(r-1)}$. Hence, the sequence $\{w_n^{(p)}\}_{n\geq 0}$ satisfies Equation (3), with initial data,

$$w_0^{(p)} = \cdots = w_{p-1}^{(p)} = 0,\ \ w_p^{(p)} = 1 \text{ and } w_n^{(p)} = a_{r-p-1}v_{n-1}^{(r-1)} + w_{n-1}^{(p-1)} = 0 \text{ for } p+1\leq n\leq r-1.$$

Hence, by applying Lemma 2.1, we derive that the two sequences $\{w_n^{(p)}\}_{n\geq 0}$ and $\{v_n^{(p)}\}_{n\geq 0}$ are identical. Therefore, for every p $(0\leq p\leq r-2)$, we have $v_n^{(p)} = a_{r-p-1}v_{n-1}^{(r-1)} + a_{r-p}v_{n-2}^{(r-1)} + \cdots + a_{r-1}v_{n-p-1}^{(r-1)}$. $\square$

The result of Theorem 2.2 can be also established by induction on p. However, we have used here an elementary process based on Lemma 2.1 and the fact that $\mathcal{E}_{\mathbb{K}}^{(r)}(a_0, \ldots, a_{r-1})$ is a $\mathbb{K}$-vector space

Expression (8) shows that the sequence $\{v_n^{(r-1)}\}_{n\geq 0}$ will play a central role in the sequel. The sequence $\{v_n^{(r-1)}\}_{n\geq 0}$, is called *dynamic solution* of Equation (3).

For illustrative purpose, we propose the following special case.

Example 2.3. *For* $r = 4$, *Expression* (3) *takes the form*

$$v_n = a_0 v_{n-1} + a_1 v_{n-2} + a_2 v_{n-3} + a_3 v_{n-4} \ \textit{for} \ \ n \geq 4.$$

Therefore, the terms $v_n^{(0)}$, $v_n^{(1)}$, $v_n^{(2)}$ *and* $v_n^{(3)}$ *of the fundamental system are expressed in terms of the dynamic solution* $v_n^{(r-1)} = v_n^{(3)}$ *under the form,*

$$v_n^{(0)} = a_3 v_{n-1}^{(3)},\ v_n^{(1)} = a_2 v_{n-1}^{(3)} + a_3 v_{n-2}^{(3)},\ v_n^{(2)} = a_1 v_{n-1}^{(3)} + a_2 v_{n-2}^{(3)} + a_3 v_{n-3}^{(3)},$$

for $n \geq 4$.

2.2. **Analytical formulas of the dynamic solution: First approach.** Let λ_i $(1 \leq i \leq s)$ be the roots of the characteristic polynomial $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$, related to sequence (3), of multiplicities m_1, m_2,..., m_s, respectively. For every $m_i \geq 1$ we set $\mathcal{E}_k^{[i]} = \{(n_1, \ldots, n_s) \in \mathbb{N}^{s-1};\ \ n_1 + \cdots + n_{i-1} + n_{i+1} + \cdots + n_s = m_i - k - 1\}$. In [5, Section 4.1], the following expression was considered,

$$\gamma_k^{[i]}(\lambda_1, \ldots, \lambda_s) = (-1)^{r-m_i} \sum_{\mathcal{E}_k^{[i]}} \left(\prod_{1\leq j\neq i\leq s} \frac{\binom{n_j + m_j - 1}{n_j}}{(\lambda_j - \lambda_i)^{n_j + m_j}} \right) \tag{9}$$

for $0 \leq k \leq m_i - 1$ and $1 \leq i \leq s$.

Example 2.4. *Let* $r = 7$ *and suppose that* $s = 3$, $m_1 = 2$, $m_2 = 1$ *and* $m_3 = 4$. *For* $i = 3$, $k = 1$ *we have* $\mathcal{E}_1^{[3]} = \{(n_1, n_2);\ n_1 + n_2 = 4 - 1 - 1 = 2\}$. *Therefore, we have,*

$$\gamma_1^{[3]}(\lambda_1, \lambda_2, \lambda_3) = (-1)^3 \sum_{n_1+n_2=2} \frac{\binom{n_1 + m_1 - 1}{n_1}}{(\lambda_1 - \lambda_3)^{n_1+m_1}} \cdot \frac{\binom{n_2 + m_2 - 1}{n_2}}{(\lambda_2 - \lambda_3)^{n_2+m_2}}.$$

Therefore, since $\binom{n_2}{n_2} = 1$ *and* $\binom{n_1+1}{n_1} = n_1 + 1$, *we obtain,*

$$\gamma_1^{[3]}(\lambda_1, \lambda_2, \lambda_3) = (-1)^3 \sum_{n_1+n_2=2} \frac{n_1 + 1}{(\lambda_1 - \lambda_3)^{n_1+2}} \cdot \frac{1}{(\lambda_2 - \lambda_3)^{n_2+1}}.$$

The analytic formula of the dynamic solution $v_n^{(r-1)}$ related to the fundamental system (7), can be expressed in terms of the roots λ_i $(1 \leq i \leq s)$ of the polynomial $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$, by the previous expressions (9) of the $\gamma_k^{[i]}(\lambda_1, \ldots, \lambda_s)$. More precisely, we have the following result.

Proposition 2.5. *The analytic expression of the dynamic solution $v_n^{(r-1)}$ is given by the following formula $v_n^{(r-1)} = \sum_{i=1}^{s} \left(\sum_{k=0}^{m_i-1} \beta_{i,k}^{(r-1)} n^k \right) \lambda_i^n$, for every $n \geq r$, where*

$$\beta_{i,k}^{(r-1)} = \sum_{t=k}^{m_i-1} s(t,k) \frac{\gamma_t^{[i]}(\lambda_1, \ldots, \lambda_s)}{t! \lambda_i^t}, \tag{10}$$

and the $s(t,k)$ are the Stirling numbers of the first kind.

Indeed, from [3, Theorem 2.2], we have $v_n^{(r-1)} = \sum_{i=1}^{s} \left(\sum_{k=0}^{m_i-1} \binom{n}{k} \gamma_k^{[i]}(\lambda_1, \ldots, \lambda_s) \right) \lambda_i^{n-k}$, for all $n \geq r$. On the other hand, it is well known that $\frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1) = \sum_{t=0}^{k} s(k,t) n^t$, where the $s(k,t)$ are Stirling numbers of the first kind. Therefore, the combinatorial expression $\binom{n}{k}$ can be also expressed in terms of the Stirling numbers of the first kind as follows $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \sum_{t=0}^{k} \frac{s(k,t)}{k!} n^t$. Thus, we get $v_n^{(r-1)} = \sum_{i=1}^{s} \left(\sum_{k=0}^{m_i-1} \left(\sum_{t=0}^{k} s(k,t) n^t \right) \frac{\gamma_k^{[i]}(\lambda_1, \ldots, \lambda_s)}{k! \lambda_i^k} \right) \lambda_i^n$, for every $n \geq r$, or equivalently,

$$v_n^{(r-1)} = \sum_{i=1}^{s} \left(\sum_{k=0}^{m_i-1} \left(\sum_{t=k}^{m_i-1} s(t,k) \frac{\gamma_t^{[i]}(\lambda_1, \ldots, \lambda_s)}{t! \lambda_i^t} \right) n^k \right) \lambda_i^n, \text{ for every } n \geq r.$$

Therefore, the results follows, namely, Expression (10) is established. □

More generally, the result of Proposition 2.5 allows us to determine the expressions of $v_{n-d}^{(r-1)}$ $(1 \leq d \leq r-1)$. In summary, we have the following result.

Proposition 2.6. *Under the preceding data, for $d = 1, \ldots, r-1$, we have*

$$v_{n-d}^{(r-1)} = \sum_{i=1}^{s} \left(\sum_{j=0}^{m_i-1} C_{i,j}^{(d)} n^j \right) \lambda_i^n, \textit{ for every } n \geq r,$$

where

$$C_{i,j}^{(d)} = \lambda_i^{-d} \sum_{k=j}^{m_i-1} (-1)^{k-j} \beta_{i,k}^{(r-1)} \binom{k}{j} d^{k-j}, \tag{11}$$

such that the $\beta_{i,k}^{(r-1)}$ are as in (10)*.*

Indeed, from Proposition 2.5, we derive,

$$v_{n-d}^{(r-1)} = \sum_{i=1}^{s} \left(\sum_{k=0}^{m_i-1} \beta_{i,k}^{(r-1)} (n-d)^k \right) \lambda_i^{n-d} = \sum_{i=1}^{s} \left(\sum_{k=0}^{m_i-1} \beta_{i,k}^{(r-1)} \sum_{j=0}^{k} \binom{k}{j} n^j (-d)^{k-j} \right) \lambda_i^{n-d}.$$

Therefore, we get $v_{n-d}^{(r-1)} = \sum_{i=1}^{s}\left(\sum_{j=0}^{m_i-1}\left(\lambda_i^{-d}\sum_{k=j}^{m_i-1}\beta_{i,k}^{(r-1)}\binom{k}{j}(-1)^{k-j}d^{k-j}\right)n^j\right)\lambda_i^n$, which allows us to derive Expression (11). □

For reason of clarity, in the following corollary, we consider the special useful case, for illustrating Propositions 2.5 and 2.6.

Corollary 2.7. ***Special case*** $s = 2$***:*** $m_1 \geq 2$ ***and*** $m_2 \geq 2$***.** *Under the data of Propositions 2.5 and 2.6, suppose that $s = 2$: $m_1 \geq 2$ and $m_2 \geq 2$, then, we have,*

$$v_n^{(r-1)} = \left(\sum_{k=0}^{m_1-1}\beta_{1,k}^{(r-1)}n^k\right)\lambda_1^n + \left(\sum_{k=0}^{m_2-1}\beta_{2,k}^{(r-1)}n^k\right)\lambda_2^n,$$

for every $n \geq r = m_1 + m_2$, where the $\beta_{i,k}^{(r-1)}$, $i = 1,\ 2$ are as in (10) *, namely,*

$$\beta_{1,k}^{(r-1)} = \sum_{t=k}^{m_1-1}s(t,k)\frac{\gamma_t^{[1]}(\lambda_1,\lambda_2)}{t!\lambda_1^t} \text{ and } \beta_{2,k}^{(r-1)} = \sum_{t=k}^{m_2-1}s(t,k)\frac{\gamma_t^{[2]}(\lambda_1,\lambda_2)}{t!\lambda_2^t}, \tag{12}$$

and the $s(t,k)$ are the Stirling numbers of the first kind. Moreover, for every $d = 1,\dots,r-1$, we have,

$$v_{n-d}^{(r-1)} = \left(\sum_{j=0}^{m_1-1}C_{1,j}^{(d)}n^j\right)\lambda_1^n + \left(\sum_{j=0}^{m_2-1}C_{2,j}^{(d)}n^j\right)\lambda_2^n,$$

for every $n \geq r = m_1 + m_2$, where the $C_{1,j}^{(d)}$ are as in (11)*, namely,*

$$C_{1,j}^{(d)} = \lambda_1^{-d}\sum_{k=j}^{m_1-1}(-1)^{k-j}\beta_{1,k}^{(r-1)}\binom{k}{j}d^{k-j} \text{ and } C_{2,j}^{(d)} = \lambda_2^{-d}\sum_{k=j}^{m_2-1}(-1)^{k-j}\beta_{2,k}^{(r-1)}\binom{k}{j}d^{k-j},$$

such that the $\beta_{1,k}^{(r-1)}$ and $\beta_{2,k}^{(r-1)}$ are given by (12).

2.3. **Analytical formulas of the dynamic solution: Second approach.** For a given sequence (3), it was shown in [13] that,

$$v_n = \rho(n,r)A_0 + \rho(n-1,r)A_1 + \cdots + \rho(n-r+1,r)A_{r-1}, \tag{13}$$

for every $n \geq r$, where $A_i = a_{r-1}v_i + \cdots + a_iv_{r-1}$ $(0 \leq i \leq r-1)$ and

$$\rho(n,r) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-r}\frac{(k_0+k_1+\cdots+k_{r-1})!}{k_0!\,k_1!\cdots k_{r-1}!}\ a_0^{k_0}a_1^{k_1}\cdots a_{r-1}^{k_{r-1}}, \tag{14}$$

with $\rho(r,r) = 1$ and $\rho(n,r) = 0$ for $n \leq r-1$. Let λ_i $(1 \leq i \leq s)$ be the roots of the polynomial $P(z) = z^r - a_0z^{r-1} - \cdots - a_{r-1}$, of multiplicities m_1, m_2,..., m_s, respectively. Using the divided difference techniques and Newton interpolation method, it was established in [1, Theorem 3.1] that Expression (14) of $\rho(n,r)$, can be formulated in terms of the roots λ_i $(1 \leq i \leq s)$ and their multiplicities m_1, m_2,..., m_s as follows,

$$\rho(n,r) = \sum_{i=1}^{s}\frac{f_{i,n}^{(m_i-1)}(\lambda_i)}{(m_i-1)!},\ \text{for every } n \geq r, \tag{15}$$

with $\rho(r,r)=1$, $\rho(n,r)=0$ for $0\leq n\leq r-1$, and $f_{i,n}(x)=\dfrac{x^{n-1}}{\prod\limits_{k=1,\,k\neq i}^{s}(x-\lambda_k)^{m_k}}$, where $f_{i,n}^{(k)}(x)$ means the derivative of order k of the function $f_{i,n}$. Especially, when the roots λ_i ($1\leq i\leq s$) are simple, namely, $s=r$ and $m_1=m_2=\cdots=m_r=1$, Expression (15) takes the form,

$$\rho(n,r)=\sum_{i=1}^{s}\frac{f_{i,n}^{(m_i-1)}(\lambda_i)}{(m_i-1)!}=\sum_{i=1}^{r}\frac{\lambda_i^{n-1}}{\prod\limits_{k=1,\,k\neq i}^{r}(\lambda_i-\lambda_k)},\tag{16}$$

Expression (16) has been also established in [2, 4]. In addition, let $\{w_n\}_{n\geq 0}$ be the sequence defined by $w_n=\rho(n+1,r)$, then Expression (13) shows that $\{w_n\}_{n\geq 0}$ satisfies the recursive relation (3), and its initial conditions are $w_0=\cdots=w_{r-2}=0$ and $w_{r-1}=1$. Hence, the dynamic solution $\{v_n^{(r-1)}\}_{n\geq 0}$ and the sequence $\{w_n\}_{n\geq 0}$ satisfy the same recursive relation (3) and own the identical initial conditions. Therefore, Lemma 2.1 shows that,

$$v_n^{(r-1)}=\rho(n+1,r),\ \text{for every } n\geq 0.\tag{17}$$

Therefore, taking into account Expressions (15) and (17), we derive that the analytic formula of the dynamic solution is given by,

$$v_n^{(r-1)}=\rho(n+1,r)=\sum_{i=1}^{s}\frac{f_{i,n+1}^{(m_i-1)}(\lambda_i)}{(m_i-1)!},\ \text{for every } n\geq r,\tag{18}$$

where the function $f_{i,n+1}$ ($1\leq i\leq s$) are defined as by $f_{i,n+1}(x)=\dfrac{x^{n}}{\prod\limits_{k=1,\,k\neq i}^{s}(x-\lambda_k)^{m_k}}$.

For $m_i=1$ we have $m_i-1=0$, therefore, we get $f_{i,n+1}^{(m_i-1)}(x)=f_{i,n+1}(x)=\dfrac{x^{n}}{\prod\limits_{k=1,\,k\neq i}^{s}(x-\lambda_k)}$. For technical reasons, we set $f_{i,n+1}(x)=q_n(x)H_i(x)$, where $q_n(x)=x^n$ and $H_i(x)=\dfrac{1}{\prod\limits_{k=1,\,k\neq i}^{s}(x-\lambda_k)^{m_k}}$.

For $m_i\geq 2$, let compute the explicit formula of the derivative of order m_i-1 of the function $f_{i,n+1}(x)=\dfrac{x^{n}}{\prod\limits_{k=1,\,k\neq i}^{s}(x-\lambda_k)^{m_k}}$. To achieve our goal, we will proceed in two steps. First, let f, g two functions admitting derivatives of order $m\geq 1$ on a nonempty subset of $\mathbb{R}$. It is well known that, we have $(fg)^{(m)}=\sum\limits_{d=0}^{m}\binom{m}{d}f^{(d)}g^{(m-d)}$. Application of this former formula to $f_{i,n+1}(x)=q_n(x)H_i(x)$, we obtain,

$$f_{i,n+1}^{(m)}(x)=\sum_{d=0}^{m}\binom{m}{d}q_n^{(d)}(x)H_i^{(m-d)}(x)=\sum_{d=0}^{m}\binom{m}{d}(n)_d x^{n-d}H_i^{(m-d)}(x),$$

where $(n)_d = n(n-1)\cdots(n-d+1)$. Since $(n)_d = \sum_{h=0}^{d} s(d,h)n^h$, where the $s(d,h)$ are the Stirling numbers of the first kind, we show that,

$$f_{i,n+1}^{(m)}(x) \;=\; \sum_{d=0}^{m} \binom{m}{d} (n)_d x^{n-d} H_i^{(m-d)}(x) = \sum_{d=0}^{m} \binom{m}{d} \left(\sum_{h=0}^{d} s(d,h)n^h \right) H_i^{(m-d)}(x) x^{n-d}.$$

Using the identity $\displaystyle\sum_{d=0}^{m}\sum_{h=0}^{d} x_{d,h} = \sum_{h=0}^{m}\sum_{d=h}^{m} x_{d,h}$, for a bi-indexed sequence $x_{d,h}$, we obtain $\displaystyle f_{i,n+1}^{(m)}(x) = \sum_{h=0}^{m} \left(\sum_{d=h}^{m} s(d,h) \binom{m}{d} H_i^{(m-d)}(x) x^{-d} \right) n^h x^n$. Therefore, for $m_i \geq 2$ we set $x = \lambda_i$ and $m = m_i - 1$, thus we arrive to have,

$$f_{i,n+1}^{(m_i-1)}(\lambda_i) = \sum_{h=0}^{m_i-1} \left(\sum_{d=h}^{m_i-1} s(d,h) \binom{m_i-1}{d} H_i^{(m_i-d-1)}(\lambda_i)\lambda_i^{-d} \right) n^h \lambda_i^n. \tag{19}$$

Second, to improve Expression (19), we will give the explicit form of the $p-th$ derivative of the function $H_i(x)$. For this purpose, we use the following well-known formula,

$$(f_1 \cdot f_2 \cdots f_s)^{(m)} = \sum_{k_1+\cdots+k_s=m} \binom{m}{k_1 \; \ldots \; k_s} \prod_{j=1}^{s} f_j^{(k_j)},$$

where $\displaystyle\binom{m}{k_1 \; \ldots \; k_s} = \frac{m!}{k_1!k_2!\cdots k_s!}$ and $f_j : E \to \mathbb{R}$ $(1 \leq j \leq s)$ are functions defined on a subset $E \neq \emptyset$ of $\mathbb{R}$, which are n times differentiable, and $f^{(p)}$ is the derivative of order p of the function f. Moreover, for every integer $m' \neq 0$, the derivative of order $p \geq 1$ of the function $f(x) = (x-\lambda)^{m'}$, is given by $f^{(p)}(x) = m'(m'-1)\cdots(m'-p+1)(x-\lambda)^{m'-p}$, and when $m' = -m$ with $m \geq 1$, we get $f^{(p)}(x) = (-1)^p \dfrac{(m+p-1)!}{(m-1)!}(x-\lambda)^{-m-p}$. Now, applying the above formulas to the function $H_i(x)$, written under the form $H_i(x) = \dfrac{1}{\prod_{j=1,\, j\neq i}^{s} (x-\lambda_j)^{m_j}} = \prod_{j=1,\, j\neq i}^{s} h_j(x)$, with $h_j(x) = (x-\lambda_j)^{-m_j}$, we obtain $\displaystyle H_i^{(k)}(x) = \sum_{\hat{\varepsilon}_k^{[i]}} \binom{k}{p_1 \; \ldots \; p_s} \prod_{j=1,\, j\neq i}^{s} h_j^{(p_j)}(x)$, where $\hat{\varepsilon}_k^{[i]} = \{(p_1, p_2, ..., p_s) \in \mathbb{N}^{s-1};\; p_1 + \cdots + p_{i-1} + p_{i+1} + \cdots + p_s = k\}$. Since, $h_j^{(p_j)}(x) = (-1)^{p_j} \dfrac{(m_j+p_j-1)!}{(m_j-1)!}(x-\lambda_j)^{-m_j-p_j}$, for every j $(1 \leq j \neq i \leq s)$, we derive the following lemma.

Lemma 2.8. *For every $k \geq 1$ and $1 \leq i \leq s$, we have,*

$$H_i^{(k)}(x) = (-1)^k \sum_{\hat{\varepsilon}_k^{[i]}} \binom{k}{p_1 \ldots p_s} \prod_{j=1,\, j\neq i}^{s} \frac{(m_j+p_j-1)!}{(m_j-1)!}(x-\lambda_j)^{-m_j-p_j}. \tag{20}$$

Summarizing, through Expressions (16), (19) and (20), we can formulate the following result.

Theorem 2.9. *Let λ_i ($1 \le i \le s$) be the roots of the polynomial $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$, associated to the recursive relation* (3)*, of multiplicities m_1, m_2,..., m_s, respectively. Suppose that for every root λ_i the associated multiplicity $m_i \ge 2$ ($1 \le i \le s$). Then, the analytic formula of the dynamic solution is given as follows,*

$$v_n^{(r-1)} = \sum_{i=1}^{s} \frac{1}{(m_i - 1)!} \sum_{h=0}^{m_i-1} \left(\sum_{d=h}^{m_i-1} s(d,h) \binom{m_i - 1}{d} H_i^{(m_i-d-1)}(\lambda_i) \lambda_i^{-d} \right) n^h \lambda_i^n, \tag{21}$$

for every $n \ge r$, where $s(d,h)$ are the Stirling numbers of the first kind and $H_i(x) = \dfrac{1}{\prod\limits_{k=1,\, k \ne i}^{s} (x - \lambda_k)^{m_k}}$ and the $H_i^{(k)}(x)$ are as in (20).

Suppose that for every root λ_i the associated multiplicity $m_i = 1$ ($1 \le i \le r$). Then, the analytic formula of the dynamic solution is given as follows,

$$v_n^{(r-1)} = \sum_{i=1}^{r} \frac{\lambda_i^n}{\prod\limits_{k=1,\, k \ne i}^{r} (\lambda_i - \lambda_k)}, \tag{22}$$

for every $n \ge r$.

We illustrate Theorem 2.9 by the following special case.

Special case: $s = m_1 = m_2 = 2$. Let determine the dynamic solution $v_n^{(3)}$, when $s = m_1 = m_2 = 2$. By using Equation (21), we infer that,

$$v_n^{(3)} = \sum_{i=1}^{2} \frac{1}{1!} \sum_{h=0}^{1} \left(\sum_{d=h}^{1} s(d,h) \binom{1}{d} H_i^{(1-d)}(\lambda_i) \lambda_i^{-d} \right) n^h \lambda_i^n = \Omega_1(n) + \Omega_2(n),$$

where

$$\Omega_i(n) = \sum_{h=0}^{1} \left(\sum_{d=h}^{1} s(d,h) \binom{1}{d} H_i^{(1-d)}(\lambda_i) \lambda_i^{-d} \right) n^h \lambda_i^n = \beta_{i,0} \lambda_i^n + \beta_{i,1} n \lambda_i^n,$$

for $i = 1, 2$. Since $s(0,0) = s(1,1) = 1$ and $s(1,0) = 0$, a straightforward computation implies that $\beta_{1,0} = H_1^{(1)}(\lambda_1)$, $\beta_{1,1} = H_1^{(0)}(\lambda_1)\lambda_1^{-1}$, $\beta_{2,0} = H_2^{(1)}(\lambda_2)$, $\beta_{2,1} = H_2^{(0)}(\lambda_2)\lambda_2^{-1}$. Applying Equation (20), we obtain,

$$\beta_{1,0} = \frac{-2}{(\lambda_1 - \lambda_2)^3}, \quad \beta_{1,1} = \frac{1}{(\lambda_1 - \lambda_2)^2 \lambda_1}, \quad \beta_{2,0} = \frac{2}{(\lambda_1 - \lambda_2)^3}, \quad \beta_{2,1} = \frac{1}{(\lambda_1 - \lambda_2)^2 \lambda_2}.$$

Therefore, for $s = m_1 = m_2 = 2$, the dynamic solution $v_n^{(3)}$ takes the form,

$$v_n^{(3)} = \frac{-2}{(\lambda_1 - \lambda_2)^3} \lambda_1^n + \frac{1}{(\lambda_1 - \lambda_2)^2 \lambda_1} n \lambda_1^n + \frac{2}{(\lambda_1 - \lambda_2)^3} \lambda_2^n + \frac{1}{(\lambda_1 - \lambda_2)^2 \lambda_2} n \lambda_2^n,$$

for every $n \ge 0$.

Let λ_i ($1 \le i \le s$) be the roots of the (characteristic) polynomial $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$, associated to the recursive relation (3), of multiplicities m_1, m_2,..., m_s, respectively. Without loss of generality, we set,

$$\mathcal{Z}_1 = \{\lambda_i, \text{ root of } P(z) \text{ with } m_i = 1\}; \mathcal{Z}_2 = \{\lambda_i, \text{ root of } P(z) \text{ with } m_i \ge 2\}.$$

Then, combining the two cases (21) and (22) of Theorem 2.9, we get the following general result.

Theorem 2.10. *Let λ_i ($1 \leq i \leq s$) be the roots of the polynomial $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$, associated to the recursive relation* (3)*, of multiplicities m_1, m_2,..., m_s, respectively. Then, the analytic formula of the dynamic solution is given as follows $v_n^{(r-1)} = \Phi_n^{(r)} + \Psi_n^{(r)}$, for every $n \geq r$, where*

$$\Phi_n^{(r)} = \sum_{i \in \mathcal{Z}_1} \frac{1}{\prod\limits_{k=1,\, k\neq i}^{s} (\lambda_i - \lambda_k)^{m_k}} \lambda_i^n \text{ and } \Psi_n^{(r)} = \sum_{i \in \mathcal{Z}_2} \left(\frac{1}{(m_i-1)!} \sum_{h=0}^{m_i-1} \Delta_{i,h}^{(r)} n^h \right) \lambda_i^n,$$

where $\Delta_{i,h}^{(r)} = \sum\limits_{d=h}^{m_i-1} s(d,h) \binom{m_i-1}{d} H_i^{(m_i-d-1)}(\lambda_i) \lambda_i^{-d}$, *with* $H_i(x) = \dfrac{1}{\prod\limits_{k=1,\, k\neq i}^{s} (x - \lambda_k)^{m_k}}$,

and the $H_i^{(k)}(x)$ are as in (20).

Since the analytic expression of $v_{n-p}^{(r-1)}$, for $p = 1, \ldots, r-1$, will be useful in the sequel, the result of Theorem 2.10 allows us to obtain,

$$v_{n-p}^{(r-1)} = \Phi_{n-p}^{(r)} + \Psi_{n-p}^{(r)}, \text{ for every } n \geq p,$$

where

$$\Phi_{n-p}^{(r)} = \sum_{i \in \mathcal{Z}_1} \frac{\lambda_i^{n-p}}{\prod\limits_{k=1,\, k\neq i}^{s} (\lambda_i - \lambda_k)^{m_k}}, \quad \Psi_{n-p}^{(r)} = \sum_{i \in \mathcal{Z}_2} \frac{1}{(m_i-1)!} \left[\sum_{h=0}^{m_i-1} \Delta_{i,h}^{(r)} (n-p)^h \right] \lambda_i^{n-p}.$$

And a similar process used for establishing Proposition 2.6, shows that the expression $\sum\limits_{h=0}^{m_i-1} \Delta_{i,h}^{(r)} (n-p)^h = \sum\limits_{h=0}^{m_i-1} \Delta_{i,h}^{(r)} \left(\sum\limits_{k=0}^{h} \binom{h}{k} n^k (-p)^{h-k} \right)$, can be written under the form $\sum\limits_{h=0}^{m_i-1} \Delta_{i,h}^{(r)} (n-p)^h = \sum\limits_{k=0}^{m_i-1} \left(\sum\limits_{h=k}^{m_i-1} (-1)^{h-k} \binom{h}{k} \Delta_{i,h}^{(r)} p^{h-k} \right) n^k$. Therefore, we derive the following proposition.

Proposition 2.11. *Under the data of Theorem 2.10, for $1 \leq p \leq r-1$, we have,*

$$v_{n-p}^{(r-1)} = \sum_{i \in \mathcal{Z}_1} \beta_{i,p}^{(r)} \lambda_i^n + \sum_{i \in \mathcal{Z}_2} \beta_{i,k,p}^{(r)} \lambda_i^n, \text{ for } n-p \geq r$$

where $\beta_{i,p}^{(r)} = \dfrac{\lambda_i^{-p}}{\prod\limits_{k=1,\, k\neq i}^{s} (\lambda_i - \lambda_k)^{m_k}}$ *and* $\beta_{i,k,p}^{(r)} = \dfrac{\lambda_i^{-p}}{(m_i-1)!} \sum\limits_{k=0}^{m_i-1} \left(\sum\limits_{h=k}^{m_i-1} (-1)^{h-k} \binom{h}{k} \Delta_{i,h}^{(r)} p^{h-k} \right) n^k$,

such that $\Delta_{i,h}^{(r)} = \sum\limits_{d=h}^{m_i-1} s(d,h) \binom{m_i-1}{d} H_i^{(m_i-d-1)}(\lambda_i) \lambda_i^{-d}$, *and* $H_i(x) = \dfrac{1}{\prod\limits_{k=1,\, k\neq i}^{s} (x - \lambda_k)^{m_k}}$,

where the $H_i^{(k)}(x)$ are as in (20).

For illustrative purpose of Theorems 2.9 and 2.10, we consider the special useful two cases: $s=2$ with $m_1 \geq 2$, $m_2 \geq 2$ and $s=3$ with $m_1=m_2=1$, $m_3 \geq 2$. For the first case we show that $\mathcal{Z}_1=\emptyset$ and $\mathcal{Z}_2=\{1,\, 2\}$. Thus, we have the following first corollary.

Corollary 2.12. ***Special case*** $s=2$***:*** $m_1 \geq 2$ ***and*** $m_2 \geq 2$**.** *Under the data of Theorem 2.10, we have,*

$$v_{n-p}^{(r-1)}=\frac{\lambda_1^{-p}}{(m_1-1)!}\sum_{k=0}^{m_1-1}\left(\sum_{h=k}^{m_1-1}\gamma_1(h,k)\right)n^k\lambda_1^n+\frac{\lambda_2^{-p}}{(m_2-1)!}\sum_{k=0}^{m_2-1}\left(\sum_{h=k}^{m_2-1}\gamma_2(h,k)\right)n^k\lambda_2^n,$$

for every $n-p \geq r$, *with* $1 \leq p \leq r-1$, *where* $\gamma_1(h,k)=(-1)^{h-k}\binom{h}{k}\Delta_{1,h}^{(r)}p^{h-k}$ *and* $\gamma_2(h,k)=(-1)^{h-k}\binom{h}{k}\Delta_{2,h}^{(r)}p^{h-k}$, *with* $\Delta_{1,h}^{(r)}=\sum_{d=h}^{m_1-1}s(d,h)\binom{m_1-1}{d}H_1^{(m_1-d-1)}(\lambda_1)\lambda_1^{-d}$, $\Delta_{2,h}^{(r)}=\sum_{d=h}^{m_2-1}s(d,h)\binom{m_2-1}{d}H_2^{(m_2-d-1)}(\lambda_2)\lambda_2^{-d}$, $H_i(x)=\dfrac{1}{\prod_{k=1,\,k\neq i}^{s}(x-\lambda_k)^{m_k}}$, *and the* $H_i^{(k)}(x)$ *are as in* (20).

For the second, we show that $\mathcal{Z}_1=\{1,\, 2\}$ and $\mathcal{Z}_2=\{3\}$. Thus, we have the following corollary,

Corollary 2.13. ***Special case*** $s=3$ ***with*** $m_1=m_2=1$**,** $m_3 \geq 2$**.** *Under the data of Theorem 2.10, we have,*

$$v_{n-p}^{(r-1)}=\frac{\lambda_1^{-p}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)^{m_3}}\lambda_1^n+\frac{\lambda_2^{-p}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)^{m_3}}\lambda_2^n+\Omega_3(n),$$

for $n-p \geq r$, *with* $1 \leq p \leq r-2$, *where* $\Omega_3(n)=\dfrac{\lambda_3^{-p}}{(m_3-1)!}\sum_{k=0}^{m_3-1}\left(\sum_{h=k}^{m_3-1}\gamma_3(h,k)\right)n^k\lambda_3^n$, $\gamma_3(h,k)=(-1)^{h-k}\binom{h}{k}\Delta_{3,h}^{(r)}p^{h-k}$, *with* $\Delta_{3,h}^{(r)}=\sum_{d=h}^{m_3-1}s(d,h)\binom{m_3-1}{d}H_3^{(m_3-d-1)}(\lambda_3)\lambda_3^{-d}$ *and* $H_3(x)=\dfrac{1}{\prod_{k=1}^{2}(x-\lambda_k)}$, *such that the* $H_i^{(k)}(x)$ *are as in* (20).

3. Construction of the inverse of generalized Vandermonde matrix via the analytic formula (5)

We have exhibited the close relation between the analytical form of sequences (3) and the generalized Vandermonde systems, through the equivalence of the two generalized Vandermonde systems (2) and (6). This section is devoted to the process of constructing the inverse of the generalized Vandermonde matrix, using the analytic formula (5) of the elements $\{v_n^{(p)}\}_{n\geq 0}$ of the fundamental systems (7).

We first introduce some useful notations, allowing us to introduce the generalized Vandermonde matrix and to study its inverse. Let $\mathbb{C}$-vector space of the polynomials $\mathbb{C}[z]$, and consider the derivation degree $Dp(z) = z\dfrac{dp}{dz}(z)$. Let p_0, p_1, ..., p_{r-1} be in $\mathbb{C}[z]$ and $f : \mathbb{C} \to \mathbb{C}^r$, the valued vector function defined by,

$$f(z) = (p_0(z), p_1(z), \ldots, p_{r-1}(z))^T,$$

where $(\gamma_0, \gamma_1, \ldots, \gamma_{r-1})^T$ denotes a vector column. We extend the derivation degree to the preceding vector function as follows,

$$Df(z) = (Dp_0(z), Dp_1(z), \ldots, Dp_{r-1}(z))^T.$$

More generally, we have $D^{(k)}f(z) = (D^{(k)}p_0(z), D^{(k)}p_1(z), \ldots, D^{(k)}p_{r-1}(z))^T$, for every $k \geq 0$, where $D^{(0)} = 1_d$, $D^{(1)} = D$, ..., $D^{(k)} = Do \ldots oD$, k times, for $k \geq 2$.

Let $\lambda_1, \ldots, \lambda_s$ be non-zero distinct s complex or real numbers, and s integers $m_1, \ldots, m_s$, with $r = m_1 + \cdots + m_s$. Let $C : \mathbb{C} \to \mathbb{C}^r$ be the valued vector function defined by,

$$C(z) = (1, z, \ldots, z^{r-1})^T, \text{ where } z \in \mathbb{C}.$$

For every $k \geq 0$, we consider the family of vector columns,

$$\begin{cases} c_i = C(\lambda_i) = (1, \lambda_i, \lambda_i^2, \ldots, \lambda_i^{r-1})^T \text{ for } k = 0 \\ c_i^{(k)} = D^{(k)}C(\lambda_i) = (0, \lambda_i, 2^k\lambda_i^2, \ldots, (r-1)^k\lambda_i^{r-1})^T \text{ for } k \geq 1. \end{cases} \tag{23}$$

The *generalized Vandermonde matrix* of order r associated to the preceding family of vectors column (23), is given by,

$$\mathbb{V} = [c_1, c_1^{(1)}, \ldots, c_1^{(m_1-1)}, \ldots, c_s, c_s^{(1)}, \ldots, c_s^{(m_s-1)}], \tag{24}$$

According to [9], we have,

$$\det \mathbb{V} = \left(\prod_{i=1}^{s} \lambda_i^{m_i(m_i-1)/2}\right)\left(\prod_{i=1}^{s}[0!1!\ldots(m_i-1)!]\right)\left(\prod_{j>i}(\lambda_j - \lambda_i)^{m_j m_i}\right).$$

This expression shows that $\det \mathbb{V} \neq 0$, because each $\lambda_i \neq 0$. Then, the generalized Vandermonde matrix $\mathbb{V}$ has inverse $\mathbb{V}^{-1}$.

Let built the process of the inversion of the generalized Vandermonde matrix (24), by utilizing the analytic formula (5) of the sequences of the fundamental system (7). For reason of clarity and conciseness, let consider the following useful notations of the two vectors column,

$$\mathbb{B} = (\beta_{1,0}, \ldots, \beta_{1,m_1-1}, \ldots, \beta_{s,0}, \ldots, \beta_{s,m_s-1})^T \text{ and } \Delta = (\alpha_0, \alpha_1, \ldots, \alpha_{r-1})^T,$$

where the $\beta_{i,j}$ ($1 \leq i \leq s$, $0 \leq j \leq m_i - 1$) are the scalars of the analytic formula (5) and $\alpha_0, \alpha_1, \ldots, \alpha_{r-1}$ are the initial data given of sequence (3). Therefore, the linear system (5) can be written under the matrix equation,

$$\mathbb{V} \cdot \mathbb{B} = \Delta. \tag{25}$$

Since the generalized Vandermonde matrix $\mathbb{V}$ is invertible, we derive,

$$\mathbb{B} = \mathbb{V}^{-1} \cdot \Delta.$$

For every p, $n = 0, \ldots, r-1$, we set $\Delta_p = (0, \ldots, 0, 1, 0, \ldots, 0)^T$ where 1 is located at the $(p+1) - th$ position, or equivalently $\Delta_p = (\alpha_0, \alpha_1, \ldots, \alpha_{r-1})^T$, whith $\alpha_n = \delta_{p,n}$. Then, the

$(p+1)$-column of the inverse $\mathbb{V}^{-1}$, of the generalized Vandermonde matrix (24) is given by,

$$\mathbb{V}_{p+1}^{(-1)} = \mathbb{V}^{-1} \cdot \Delta_p \tag{26}$$

Furthermore, the scalar components of this column vector $\mathbb{V}_{p+1}^{(-1)}$ are the solution of the linear system (5), defining the analytic formula of the linear recursive sequence (7), whose initial conditions are the entries of vector column Δ_p. More precisely, for the sequence $\{v_n^{(p)}\}_{n\geq 0}$, defined as in (7), and whose initial conditions are the entries of the column vector Δ_p, the analytic formula is,

$$v_n^{(p)} = \sum_{i=1}^{s} \left(\sum_{j=0}^{m_i-1} \beta_{i,j}^{(p)} n^j \right) \lambda_i^n.$$

And by considering the vector column,

$$\mathbb{B}^{(p)} = (\beta_{1,0}^{(p)}, \ldots, \beta_{1,m_1-1}^{(p)}, \ldots, \beta_{s,0}^{(p)}, \ldots, \beta_{s,m_s-1}^{(p)})^T,$$

we show that Expression (26) implies that the $(p+1)-th$ column, of the matrix $\mathbb{V}^{-1}$ inverse of the generalized Vandermonde matrix, is given by,

$$\mathbb{V}_{p+1}^{(-1)} = \mathbb{B}^{(p)} = (\beta_{1,0}^{(p)}, \ldots, \beta_{1,m_1-1}^{(p)}, \ldots, \beta_{s,0}^{(p)}, \ldots, \beta_{s,m_s-1}^{(p)})^T$$

Therefore, the analytic formulas of the sequences $\{v_n^{(p)}\}_{n\geq 0}$ $(0 \leq p \leq r-1)$, defined as in (7), provides us the columns of the inverse of generalized Vandermonde matrix $\mathbb{V}$ given as in (24). In summary, we formulate the following fundamental result.

Theorem 3.1. *Let $\mathbb{V}$ be the generalized Vandermande matrix* (24)*, related to the linear system* (5)*, through the matrix Equation* (25)*, namely, $\mathbb{V} \cdot \mathbb{B} = \Delta$, where $\mathbb{B}$ is the vector column $\mathbb{B} = (\beta_{1,0} \ldots, \beta_{1,m_1-1}, \ldots, \beta_{s,0}, \ldots, \beta_{s,m_s-1})^T$ obtained from the analytic expression* (5) *and $\Delta = (\alpha_0, \alpha_1, \ldots, \alpha_{r-1})^T$ is the vector related to the initial conditions of the sequence* (3)*. Then, the inverse of the generalized Vandermande matrix is given by,*

$$\mathbb{V}^{-1} = [\mathbb{B}^{(0)}, \mathbb{B}^{(1)}, \ldots, \mathbb{B}^{(r-1)}],$$

where $\mathbb{B}^{(p)} = (\beta_{1,0}^{(p)}, \ldots, \beta_{1,m_1-1}^{(p)}, \ldots, \beta_{s,0}^{(p)}, \ldots, \beta_{s,m_s-1}^{(p)})^T$ is the vector column $\mathbb{V}_{p+1}^{(-1)} = \mathbb{V}^{-1} \cdot \Delta_p$, with $\Delta_p = (\alpha_0, \alpha_1, \ldots, \alpha_{r-1})^T$ with $\alpha_n = \delta_{p,n}$, namely, $\mathbb{B}^{(p)} = \mathbb{V}_{p+1}^{(-1)} = \mathbb{V}^{-1} \cdot \Delta_p$.

Taking into account Theorem 2.2 and Theorem 3.1, as well as the results of Section 2, concerning the explicit analytic form the dynamic solution $v_n^{(r-1)}$, we will give an explicit form of the vector column $\mathbb{B}^{(p)}$. Thus, we can establish an explicit form of $\mathbb{V}^{-1}$ the inverse of the Vandermonde matrix $\mathbb{V}$.

In order to better grasp our process, let consider the special case of $s = 2$. Suppose that the characteristic polynomial of the sequence (3) is given by $P(z) = (z-\lambda_1)^{m_1}(z-\lambda_2)^{m_2}$, for the sake of generality we take $m_1, m_2 \geq 2$. First, the related generalized Vandermonde matrix of order $r = m_1 + m_2$, is given by,

$$\mathbb{V} = [c_1, c_1^{(1)}, \ldots, c_1^{(m_1-1)}, c_2, c_2^{(1)}, \ldots, c_2^{(m_2-1)}], \tag{27}$$

where the c_i ($i = 1,\, 2$), $c_1^{(k)}$ ($1 \leq k \leq m_1 - 1$) and $c_2^{(k)}$($1 \leq k \leq m_2 - 1$) are defind as in Expressions (23).

Second, the analytic expression of each element $\{v_n^{(p)}\}_{n\geq 0}$, where $0 \leq p \leq r-1$, of the associated fundamental system (7), can be written under the form,

$$v_n^{(p)} = \left(\sum_{j=0}^{m_1-1} \beta_{1,j}^{(p)} n^j\right) \lambda_1^n + \left(\sum_{j=0}^{m_2-1} \beta_{2,j}^{(p)} n^j\right) \lambda_2^n, \text{ for every } n \geq 0.$$

Third, the associated column vectors $\mathbb{B}^{(p)}$, where $0 \leq p \leq r-1$, related to the former analytic formula of $\{v_n^{(p)}\}_{n\geq 0}$, are given by,

$$\mathbb{B}^{(p)} = (\beta_{1,0}^{(p)}, \ldots, \beta_{1,m_1-1}^{(p)}, \beta_{2,0}^{(p)}, \ldots, \beta_{2,m_2-1}^{(p)})^T. \tag{28}$$

In summary, we have the proposition.

Proposition 3.2. *Under the preceding data, the inverse of the generalized Vandermonde matrix* (27) *is given by,*

$$\mathbb{V}^{-1} = [\mathbb{B}^{(0)}, \mathbb{B}^{(1)}, \ldots, \mathbb{B}^{(r-1)}],$$

where the $\mathbb{B}^{(p)}$ *are as* (28).

The previous fourth steps investigated to exemplify the preceding special case $s = 2$, allow us to formulate the following general algorithm for constructing the inverse of the generalized Vandermonde matrix,

Step 1. The generalized Vandermonde matrix. Let $\lambda_1,\, \ldots,\, \lambda_s$ be in $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), and the integer $m_1, \ldots, m_s \geq 1$ and set $r = m_1 + \cdots + m_s$. The associated generalized Vandermonde matrix of order r, is given by,

$$\mathbb{V} = [c_1, c_1^{(1)}, \ldots, c_1^{(m_1-1)}, \ldots, c_s, c_s^{(1)}, \ldots, c_s^{(m_s-1)}],$$

where the c_i ($1 \leq i \leq s$) and $c_i^{(k)}$ ($1 \leq i \leq s$, $1 \leq k \leq m_i - 1$) are the vector columns defined by (23).

Step 2. Analytic form of the associated fundamental system. Let $\{v_n^{(p)}\}_{n\geq 0}$ ($0 \leq p \leq r-1$) be the fundamental system (7), whose characteristic polynomial is $P(z) = \prod_{i=1}^{s}(z - \lambda_i)^{m_i} = z^r - a_0 z^{r-1} - \cdots - a_{r-1}$. Suppose that, for each p ($0 \leq p \leq r-1$) the analytic formula of $\{v_n^{(p)}\}_{n\geq 0}$ is computed under the form $v_n^{(p)} = \sum_{i=1}^{s}\left(\sum_{j=0}^{m_i-1} \beta_{i,j}^{(p)} n^j\right) \lambda_i^n$.

Step 3. Vectors column of $\mathbb{V}^{-1}$. For each p ($0 \leq p \leq r-1$), we consider the vectors column $\mathbb{B}^{(p)} = (\beta_{1,0}^{(p)}, \ldots, \beta_{1,m_1-1}^{(p)},\ \ldots,\ \beta_{s,0}^{(p)}, \ldots, \beta_{s,m_s-1}^{(p)})^T$.

Step 4. The inverse of the generalized Vandermonde matrix $\mathbb{V}$. The matrix $\mathbb{V}^{-1}$ is given by $\mathbb{V}^{-1} = [\mathbb{B}^{(0)}, \mathbb{B}^{(1)}, \ldots, \mathbb{B}^{(r-1)}]$.

Special case: $r = 3$ **and** $s = 2$**.** Let λ_1, λ_2 be in $\mathbb{K}$ and the integers $m_1 = 1$, $m_2 = 2$. Hence, we have $r = m_1 + m_2 = 3$. In this case, we have $c_1 = (1, \lambda_1, \lambda_1^2)^T$, $c_2 = (1, \lambda_2, \lambda_2^2)^T$ and $c_2^{(1)} = (0, \lambda_2, 2\lambda_2^2)^T$. Therefore, the associated generalized Vandermonde matrix is,

$$\mathbb{V} = \begin{bmatrix} 1 & 1 & 0 \\ \lambda_1 & \lambda_2 & \lambda_2 \\ \lambda_1^2 & \lambda_2^2 & 2\lambda_2^2 \end{bmatrix}.$$

The characteristic polynomial associated to $\mathbb{V}$ is given by

$$P(z) = (z - \lambda_1)(z - \lambda_2)^2 = z^3 - a_0 z^2 - a_1 z - a_2.$$

Let $\{v_n^{(p)}\}_{n \geq 0}$ $(0 \leq p \leq 2)$ be the fundamental system (7) related to the recursive relation $v_n = a_0 v_{n-1} + a_1 v_{n-2} + a_2 v_{n-3}$, for $n \geq 3$. Suppose that the analytic formula of each $\{v_n^{(p)}\}_{n \geq 0}$ is given by, $v_n^{(p)} = \displaystyle\sum_{i=1}^{2} \left(\sum_{j=0}^{m_i - 1} \beta_{i,j}^{(p)} n^j \right) \lambda_i^n$. Then, the inverse $\mathbb{V}^{-1}$ of the matrix $\mathbb{V}$ is as follows,

$$\mathbb{V}^{-1} = \begin{bmatrix} \beta_{1,0}^{(0)} & \beta_{1,0}^{(1)} & \beta_{1,0}^{(2)} \\ \beta_{2,0}^{(0)} & \beta_{2,0}^{(1)} & \beta_{2,0}^{(2)} \\ \beta_{2,1}^{(0)} & \beta_{2,1}^{(1)} & \beta_{2,1}^{(2)} \end{bmatrix}. \tag{29}$$

4. The explicit analytic formula for the inverse of the generalized Vandermonde matrix

4.1. **First approach.** The first approach of the dynamic solution of Subsection 2.2, allows us to exhibit explicit formulas for the entries of the inverse of the generalized Vandermonde matrix, using Propositions 2.5, 2.6 and Theorem 3.1. To reach our goal, we consider the following lemma, derived from Expression (8).

Lemma 4.1. *Let $\{v_n^{(p)}\}_{n \geq 0}$ $(p = 0, \ldots, r-2)$ be the sequence of the fundamental system* (7)*. Then, we have $v_n^{(p)} = \displaystyle\sum_{i=1}^{s} \left(\sum_{j=0}^{m_i - 1} \beta_{i,j}^{(p)} n^j \right) \lambda_i^n$, for every $n \geq 0$, where the $\beta_{i,j}^{(p)}$ are written under the form,*

$$\beta_{i,j}^{(p)} = a_{r-p-1} C_{i,j}^{(1)} + a_{r-p} C_{i,j}^{(2)} + \cdots + a_{r-1} C_{i,j}^{(p+1)}, \tag{30}$$

such that the $C_{i,j}^{(k)}$ are given as in (11)*.*

Theorem 2.2 implies that $v_n^{(p)} = a_{r-p-1} v_{n-1}^{(r-1)} + a_{r-p} v_{n-2}^{(r-1)} + \cdots + a_{r-1} v_{n-p-1}^{(r-1)}$, moreover, Proposition 2.6 shows that $v_{n-k}^{(r-1)} = \displaystyle\sum_{i=1}^{s} \left(\sum_{j=0}^{m_i - 1} C_{i,j}^{(k)} n^j \right) \lambda_i^n$, for $1 \leq k \leq p+1$, where the $C_{i,j}^{(k)}$ are given as in (11). Therefore, we obtain,

$$v_n^{(p)} = \sum_{i=1}^{s} \left(\sum_{j=0}^{m_i - 1} \left(a_{r-p-1} C_{i,j}^{(1)} + a_{r-p} C_{i,j}^{(2)} + \cdots + a_{r-1} C_{i,j}^{(p+1)} \right) n^j \right) \lambda_i^n,$$

for every $p = 0, \ldots, r-2$. Hence, the result follows. $\square$

By means of Theorem 3.1 and Lemma 4.1, we can exhibit the first result concerning the explicit expressions for the entries of the inverse of a generalized Vandermonde matrix.

Theorem 4.2. Inverse of the generalized Vandermonde matrix $\mathbb{V}$**.** *Let $\lambda_1, \dots, \lambda_s$ be non-zero distinct s real (or complex) numbers and s integers $m_1, \dots, m_s$. Let $\mathbb{V}$ be the associated generalized Vandermonde matrix* (24) *of order $r = m_1 + \cdots + m_s$. Then, the inverse $\mathbb{V}^{-1}$ of the matrix $\mathbb{V}$, is given by,*

$$\mathbb{V}^{-1} = [\mathbb{B}^{(0)}, \mathbb{B}^{(1)}, \dots, \mathbb{B}^{(r-1)}],$$

such that $\mathbb{B}^{(p)} = (\beta^{(p)}_{1,0}, \dots, \beta^{(p)}_{1,m_1-1}, \dots, \beta^{(p)}_{s,0}, \dots, \beta^{(p)}_{s,m_s-1})^T$ $(0 \leq p \leq r-1)$, namely, the explicit formulas for the entries b_{uv} $(u, v = 1, \dots, r)$ of $\mathbb{V}^{-1} = [b_{uv}]_{1 \leq u,v \leq r}$ are as follows,

$$b_{uv} = \begin{cases} \beta^{(v-1)}_{1,u-1}, & \text{for} \quad 1 \leq u \leq m_1 \\ \beta^{(v-1)}_{2,u-(m_1+1)}, & \text{for} \quad m_1+1 \leq u \leq m_1+m_2 \\ \quad \vdots & \\ \beta^{(v-1)}_{s,u-(m_1+m_2+\cdots+m_{s-1}+1)}, & \text{for } m_1+m_2+\cdots+m_{s-1}+1 \leq u \leq m_1+m_2+\cdots+m_s \end{cases}$$

where the $\beta^{(r-1)}_{i,j}$ are as in Expressions (9)-(10) *and the $\beta^{(p)}_{i,j}$ $(p = 0, \dots, r-2)$ are as in Expressions* (30)*, with the $C^{(k)}_{i,j}$ are given by* (11)*.*

For illustrating the result the former Theorem 4.2, we consider the following special case.

Special case: $r = 3$ **and** $s = 2$**.** Let λ_1, λ_2 be in two distinct complex numbers and the two integer $m_1 = 1$, $m_2 = 2$. Let $\mathbb{V} = [c_1, c_2, c^{(1)}_2]$ be the associated generalized Vandermonde matrix of order $r = m_1 + m_2 = 3$ defined by (23)-(24). Let consider the polynomial of degree 3 is given by,

$$P(z) \;=\; (z-\lambda_1)(z-\lambda_2)^2 = z^3 - (2\lambda_2+\lambda_1)z^2 + (\lambda_2^2 + 2\lambda_1\lambda_2)z - \lambda_1\lambda_2^2.$$

Thus, we have $P(z) = z^3 - a_0 z^2 - a_1 z - a_2$, with $a_0 = 2\lambda_2 + \lambda_1$, $a_1 = -(\lambda_2^2 + 2\lambda_1\lambda_2)$ and $a_2 = \lambda_1\lambda_2^2$. Let calculate the entries of $\mathbb{V}^{-1}$ the inverse of the generalized Vandermonde matrix $\mathbb{V}$. Taking into account $s(0,0) = s(1,1) = 1$ and $s(1,0) = 0$ and Equation (10) it follows

$$\beta^{(2)}_{1,0} = \gamma^{[1]}_0(\lambda_1, \lambda_2), \;\; \beta^{(2)}_{2,0} = \gamma^{[2]}_0(\lambda_1, \lambda_2) \text{ e } \beta^{(2)}_{2,1} = \frac{1}{\lambda_2}\gamma^{[2]}_1(\lambda_1, \lambda_2)$$

And, using the formula (9) we derive the $\gamma^{[1]}_0(\lambda_1, \lambda_2) = \dfrac{1}{(\lambda_2-\lambda_1)^2}$, $\gamma^{[2]}_0(\lambda_1, \lambda_2) = \dfrac{-1}{(\lambda_1-\lambda_2)^2}$ and $\gamma^{[2]}_1(\lambda_1, \lambda_2) = \dfrac{-1}{\lambda_1-\lambda_2}$. Therefore, we reach the formulas,

$$\beta^{(2)}_{1,0} = \frac{1}{(\lambda_2-\lambda_1)^2}, \;\; \beta^{(2)}_{2,0} = \frac{-1}{(\lambda_2-\lambda_1)^2} \text{ and } \beta^{(2)}_{2,1} = \frac{-1}{(\lambda_1-\lambda_2)\lambda_2}.$$

Equation (30) shows that $\beta^{(1)}_{i,j} = a_1 C^{(1)}_{i,j} + a_2 C^{(2)}_{i,j}$ and $\beta^{(0)}_{i,j} = a_2 C^{(1)}_{i,j}$. Since $a_1 = -(\lambda_2^2 + 2\lambda_1\lambda_2)$ and $a_2 = \lambda_1\lambda_2^2$, then, a direct computation, applying the formula (11), allows us to establish,

$$\beta^{(1)}_{1,0} = \frac{-2\lambda_2}{(\lambda_2-\lambda_1)^2}, \;\; \beta^{(1)}_{2,0} = \frac{2\lambda_2}{(\lambda_2-\lambda_1)^2}, \;\; \beta^{(1)}_{2,1} = \frac{\lambda_2+\lambda_1}{\lambda_2(\lambda_1-\lambda_2)},$$

$$\beta_{1,0}^{(0)}=\frac{\lambda_2^2}{(\lambda_2-\lambda_1)^2},\quad \beta_{2,0}^{(0)}=\frac{-2\lambda_1\lambda_2+\lambda_1^2}{(\lambda_1-\lambda_2)^2},\beta_{2,1}^{(0)}=\frac{-\lambda_1}{\lambda_1-\lambda_2}$$

Therefore, using the expression (29), we derive that the inverse $\mathbb{V}^{-1}$ of $\mathbb{V}=[c_1,c_2,c_2^{(1)}]$, is as follows,

$$\mathbb{V}^{-1}=\begin{bmatrix}\dfrac{\lambda_2^2}{(\lambda_2-\lambda_1)^2} & \dfrac{-2\lambda_2}{(\lambda_2-\lambda_1)^2} & \dfrac{1}{(\lambda_2-\lambda_1)^2}\\ \dfrac{-2\lambda_1\lambda_2+\lambda_1^2}{(\lambda_2-\lambda_1)^2} & \dfrac{2\lambda_2}{(\lambda_2-\lambda_1)^2} & \dfrac{-1}{(\lambda_2-\lambda_1)^2}\\ \dfrac{-\lambda_1}{\lambda_1-\lambda_2} & \dfrac{\lambda_2+\lambda_1}{\lambda_2(\lambda_1-\lambda_2)} & \dfrac{-1}{\lambda_2(\lambda_1-\lambda_2)}\end{bmatrix}.$$

4.2. **Second approach.** Let consider the second approach of the dynamic solution of Subsection 2.3, for establishing the inverse of the generalized Vandermonde matrix, using results of Theorems 2.9, 2.10 and Theorem 3.1. To reach our goal, let consider the following theorem, derived from Expression (8).

In order to identify the explicit formulas of the entries of the inverse matrix of a generalized Vandermonde matrix (24), let recall hereafter, the result of Theorem 2.9 concerning the dynamic solution (21), which can be adequate for its application in this subsection.

Lemma 4.3. *The expression of the dynamic solution $v_n^{(r-1)}$ is given by the following formula,* $v_n^{(r-1)}=\sum_{i=1}^{s}\left(\sum_{h=0}^{m_i-1}\hat{\beta}_{i,h}^{(r-1)}n^h\right)\lambda_i^n$, *for every $n\geq r$, where*

$$\hat{\beta}_{i,h}^{(r-1)}=\frac{1}{(m_i-1)!}\sum_{d=h}^{m_i-1}s(d,h)\binom{m_i-1}{d}H_i^{(m_i-d-1)}(\lambda_i)\lambda_i^{-d}, \tag{31}$$

$s(d,h)$ are the Stirling number of the first kind and $H_i^{(m_i-d-1)}(\lambda_i)$ are as in (20).

Moreover, as we will make use of the formula (8), we have also the following property.

Proposition 4.4. *For $d=1,\ldots,r-1$, we have $v_{n-d}^{(r-1)}=\sum_{i=1}^{s}\left(\sum_{j=0}^{m_i-1}\hat{C}_{i,j}^{(d)}n^j\right)\lambda_i^n$, for every $n\geq r$, where*

$$\hat{C}_{i,j}^{(d)}=\lambda_i^{-d}\sum_{k=j}^{m_i-1}(-1)^{k-j}\hat{\beta}_{i,k}^{(r-1)}\binom{k}{j}d^{k-j} \tag{32}$$

such that the $\hat{\beta}_{i,k}^{(r-1)}$ are as in (31).

In the aim to utilize Theorem 3.1 for constructing the inverse of the generalized Vandermonde matrix (24), let now consider the following lemma, derived from Expression (8).

Lemma 4.5. *Let $\{v_n^{(p)}\}_{n\geq 0}$ $(p=0,\ldots,r-2)$ be the sequence of the fundamental system* (7). *Then, for every $n\geq r$, we have $v_n^{(p)}=\sum_{i=1}^{s}\left(\sum_{j=0}^{m_i-1}\hat{\beta}_{i,j}^{(p)}n^j\right)\lambda_i^n$, such that $\hat{\beta}_{i,j}^{(p)}$ are written under the form,*

$$\hat{\beta}_{i,j}^{(p)}=a_{r-p-1}\hat{C}_{i,j}^{(1)}+a_{r-p}\hat{C}_{i,j}^{(2)}+\cdots+a_{r-1}\hat{C}_{i,j}^{(p+1)}, \tag{33}$$

where the $\hat{C}_{i,j}^{(d)}$ *are given as in* (32).

As a result of the above data, we get the following result regarding the explicit formula for the entries of the inverse of the generalized Vandermonde matrix.

Theorem 4.6. Inverse of $\mathbb{V}$**.** *Let* $\lambda_1, \ldots, \lambda_s$ *be non-zero distinct* s *complex or real numbers, and* s *integers* $m_1, \ldots, m_s$*. Let* $\mathbb{V}$ *be the associated generalized Vandermonde matrix of order* $r = m_1 + \cdots + m_s$*, thus we have*

$$\mathbb{V}^{-1} = [\hat{\mathbb{B}}^{(0)}, \hat{\mathbb{B}}^{(1)}, \ldots, \hat{\mathbb{B}}^{(r-1)}],$$

such that $\hat{\mathbb{B}}^{(p)} = (\hat{\beta}_{1,0}^{(p)}, \ldots, \hat{\beta}_{1,m_1-1}^{(p)}, \ldots, \hat{\beta}_{s,0}^{(p)}, \ldots, \hat{\beta}_{s,m_s-1}^{(p)})^T$*, where the* $\hat{\beta}_{i,j}^{(r-1)}$ *are given Expressions* (31)*, with* $H_i^{(m_i-d-1)}(\lambda_i)$ *are as in* (20)*, and the* $\hat{\beta}_{i,j}^{(p)}$ $(p = 0, \ldots, r-2)$ *are given by Expressions* (33)*, with the* $\hat{C}_{i,j}^{(d)}$ *are as in* (32)*. More precisely, the explicit formulas of the entries* $\hat{b}_{uv}$ $(u, v = 1, \ldots, r)$ *of* $\mathbb{V}^{-1} = [\hat{b}_{uv}]_{1 \le u,v \le r}$ *are as follows,*

$$\hat{b}_{uv} = \begin{cases} \hat{\beta}_{1,u-1}^{(v-1)}, & \text{for} \quad 1 \le u \le m_1 \\ \hat{\beta}_{2,u-(m_1+1)}^{(v-1)}, & \text{for} \quad m_1 + 1 \le u \le m_1 + m_2 \\ \quad \vdots & \\ \hat{\beta}_{s,u-(m_1+m_2+\cdots+m_{s-1}+1)}^{(v-1)}, & \text{for } m_1 + m_2 + \cdots + m_{s-1} + 1 \le u \le m_1 + m_2 + \cdots + m_s \end{cases}$$

where the $\hat{\beta}_{i,j}^{(r-1)}$ *are given Expressions* (31)*, with* $H_i^{(m_i-d-1)}(\lambda_i)$ *are as in* (20) *and* $\hat{\beta}_{i,j}^{(p)}$ $(p = 0, \ldots, r-2)$ *are given by Expressions* (33)*, with the* $\hat{C}_{i,j}^{(d)}$ *are as in* (32).

In order to better understand the general case of Theorem 4.6, we present below, a significant special case.

Special case: $r = 4$ **and** $s = m_1 = m_2 = 2$**.** Let λ_1, λ_2 be in two distinct complex numbers and the two integer $m_1 = 2$, $m_2 = 2$. Let $\mathbb{V} = [c_1, c_1^{(1)}, c_2, c_2^{(1)}]$ be the associated generalized Vandermonde matrix of order $r = m_1 + m_2 = 4$ defined by (23)-(24), namely,

$$\mathbb{V} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ \lambda_1 & \lambda_1 & \lambda_2 & \lambda_2 \\ \lambda_1^2 & 2\lambda_1^2 & \lambda_2^2 & 2\lambda_2^2 \\ \lambda_1^3 & 3\lambda_1^3 & \lambda_2^3 & 3\lambda_2^3 \end{bmatrix}.$$

Consider the polynomial of degree 4 given by,

$$P(z) \quad = \quad (z - \lambda_1)^2 (z - \lambda_2)^2 = z^4 - a_0 z^3 - a_1 z^2 - a_2 z - a_3$$

where $a_0 = 2(\lambda_1 + \lambda_2)$, $a_1 = -(\lambda_2^2 + 4\lambda_1\lambda_2 + \lambda_1^2)$, $a_2 = 2(\lambda_2^2\lambda_1 + \lambda_1^2\lambda_2)$ and $a_3 = -(\lambda_1^2\lambda_2^2)$. Using result of the special case illustrating Theorem 2.9, we have,

$$\hat{\beta}_{1,0}^{(3)} = \frac{-2\lambda_1\lambda_2}{(\lambda_1 - \lambda_2)^3 \lambda_1\lambda_2}; \hat{\beta}_{1,1}^{(3)} = \frac{\lambda_1\lambda_2 - \lambda_2^2}{(\lambda_1 - \lambda_2)^3 \lambda_1\lambda_2}; \hat{\beta}_{2,0}^{(3)} = \frac{2\lambda_1\lambda_2}{(\lambda_1 - \lambda_2)^3 \lambda_1\lambda_2}; \hat{\beta}_{2,1}^{(3)} = \frac{\lambda_1^2 - \lambda_1\lambda_2}{(\lambda_1 - \lambda_2)^3 \lambda_1\lambda_2}.$$

From Equation (32) and Equation (33), and because $a_1 = -(\lambda_2^2 + 4\lambda_1\lambda_2 + \lambda_1^2)$, $a_2 = 2(\lambda_2^2\lambda_1 + \lambda_1^2\lambda_2)$ and $a_3 = -(\lambda_1^2\lambda_2^2)$, a direct computation shows that,

$$\hat{\beta}_{1,0}^{(0)} = \frac{3\lambda_1^2\lambda_2^3 - \lambda_1\lambda_2^4}{(\lambda_1 - \lambda_2)^3 \lambda_1\lambda_2}; \hat{\beta}_{1,1}^{(0)} = \frac{-\lambda_1^2\lambda_2^3 + \lambda_1\lambda_2^4}{(\lambda_1 - \lambda_2)^3 \lambda_1\lambda_2}; \hat{\beta}_{2,0}^{(0)} = \frac{-3\lambda_1^3\lambda_2^2 + \lambda_1^4\lambda_2}{(\lambda_1 - \lambda_2)^3 \lambda_1\lambda_2};$$

$$\hat{\beta}_{2,1}^{(0)} = \frac{-\lambda_1^4\lambda_2 + \lambda_1^3\lambda_2^2}{(\lambda_1-\lambda_2)^3\lambda_1\lambda_2}; \hat{\beta}_{1,0}^{(1)} = \frac{-6\lambda_1^2\lambda_2^2}{(\lambda_1-\lambda_2)^3\lambda_1\lambda_2}; \hat{\beta}_{1,1}^{(1)} = \frac{-\lambda_1\lambda_2^3 + 2\lambda_1^2\lambda_2^2 - \lambda_2^4}{(\lambda_1-\lambda_2)^3\lambda_1\lambda_2};$$
$$\hat{\beta}_{2,0}^{(1)} = \frac{6\lambda_1^2\lambda_2^2}{(\lambda_1-\lambda_2)^3\lambda_1\lambda_2}; \hat{\beta}_{2,1}^{(1)} = \frac{\lambda_1^3\lambda_2 - 2\lambda_1^2\lambda_2^2 + \lambda_1^4}{(\lambda_1-\lambda_2)^3\lambda_1\lambda_2}; \hat{\beta}_{1,0}^{(2)} = \frac{3\lambda_1\lambda_2^2 + 3\lambda_1^2\lambda_2}{(\lambda_1-\lambda_2)^3\lambda_1\lambda_2};$$
$$\hat{\beta}_{1,1}^{(2)} = \frac{-\lambda_1\lambda_2^2 + 2\lambda_2^3 - \lambda_1^2\lambda_2}{(\lambda_1-\lambda_2)^3\lambda_1\lambda_2}; \hat{\beta}_{2,0}^{(2)} = \frac{-3\lambda_1\lambda_2^2 - 3\lambda_1^2\lambda_2}{(\lambda_1-\lambda_2)^3\lambda_1\lambda_2}; \hat{\beta}_{2,1}^{(2)} = \frac{\lambda_1^2\lambda_2 + \lambda_1\lambda_2^2 - 2\lambda_1^3}{(\lambda_1-\lambda_2)^3\lambda_1\lambda_2}$$

Therefore, using Theorem 3.1, we derive that $\mathbb{V}^{-1}$ the inverse of $\mathbb{V} = [c_1, c_1^{(1)}, c_2, c_2^{(1)}]$, is as follows,

$$\mathbb{V}^{-1} = \frac{1}{K}\begin{bmatrix} 3\lambda_1^2\lambda_2^3 - \lambda_1\lambda_2^4 & -6\lambda_1^2\lambda_2^2 & 3\lambda_1\lambda_2^2 + 3\lambda_1^2\lambda_2 & -2\lambda_1\lambda_2 \\ -\lambda_1^2\lambda_2^3 + \lambda_1\lambda_2^4 & -\lambda_1\lambda_2^3 + 2\lambda_1^2\lambda_2^2 - \lambda_2^4 & -\lambda_1\lambda_2^2 + 2\lambda_2^3 - \lambda_1^2\lambda_2 & \lambda_1\lambda_2 - \lambda_2^2 \\ -3\lambda_1^3\lambda_2^2 + \lambda_1^4\lambda_2 & 6\lambda_1^2\lambda_2^2 & -3\lambda_1\lambda_2^2 - 3\lambda_1^2\lambda_2 & 2\lambda_1\lambda_2 \\ -\lambda_1^4\lambda_2 + \lambda_1^3\lambda_2^2 & \lambda_1^3\lambda_2 - 2\lambda_1^2\lambda_2^2 + \lambda_1^4 & \lambda_1^2\lambda_2 + \lambda_1\lambda_2^2 - 2\lambda_1^3 & \lambda_1^2 - \lambda_1\lambda_2 \end{bmatrix},$$

where $K = (\lambda_1 - \lambda_2)^3\lambda_1\lambda_2$.

5. Analyze and discussion

In view of their numerous applications, the usual Vandermonde systems and generalized Vandermonde systems, have been largely studied in the literature (see for example [6, 7, 10, 15, 17, 20] and references therein). In order to resolve them, the topic of the inverse of associated Vandermonde and generalized Vandermonde matrices continues to attract a lot of attention, and has been the subject of numerous research papers. Therefore, various approaches have been implanted for succeeding this inversion (see for example [2, 7, 11, 15, 16, 18, 20] and references therein). Especially, the technique of LU factorization of matrix (see [12, 15, 19]), has been examined in [12], for inverting the generalized Vandermonde matrices.

In our study, we have exploited the fact the resolution of the Vandermonde and generalized Vandermonde systems (1)-(2), is linked to the determination of the analytical formula (5) of the linear recursive sequences (3). Recently, such important relation has been exploited in [2] to develop a method for inverting the Vandermonde and generalized Vandermonde matrices. More precisely, Expression (13) from [2, Theorem 2.9] has been utilized in the aim to establish some explicit formulas of the $\beta_{i,j}$ (see [2, Proposition 4.1]).

Regarding our results, we approached the inversion of the generalized Vandermonde matrix via the computation of the scalars $\beta_{i,j}$ or $\hat{\beta}_{i,j}$ ($1 \leq i \leq s$, $0 \leq j \leq m_i - 1$), by considering other explicit formulas for analytical expressions of the linear recursive sequences (3), and through another approach based on properties of linear algebra. More precisely, such explicit analytical expressions are applied to the fundamental system (7), which makes it possible to construct the vectors columns of the inverse of the generalized Vandermonde matrix $\mathbb{V}^{-1}$. This represents a new procedure, in the algorithmic construction of the matrix $\mathbb{V}^{-1}$, using Theorem 3.1. It is important to specify that, when all the roots of the polynomial $P(z) = z^r - a_0z^{r-1} - \cdots - a_{r-1}$ are simple Expression (16) and (22), permit to exhibit the inverse of generalized usual Vandermonde matrix as shown in [2, Theorem 3.3]. And by using our new process this result can be recovered easily.

6. Conclusion and perspectives

This study presents results regarding some explicit formulas for the entries of the inverse $\mathbb{V}^{-1}$ of the generalized Vandermonde matrices. Our process based on the explicit analytic formula of fundamental system (7), and an algorithmic technique, for constructing the columns of $\mathbb{V}^{-1}$. In the best of our knowledge, the formulas of the entries of $\mathbb{V}^{-1}$ are not current in the literature.

We do not claim that this study is a complete presentation of all methods for approaching the the inverse $\mathbb{V}^{-1}$ of the generalized Vandermonde matrices. However, the material presented here is a reflection of our approaches concerning the inverse $\mathbb{V}^{-1}$ of the generalized Vandermonde matrice, through the properties of fundamental system related to the associated to difference equations (3).

Finally, as there are other explicit analytic forms (5) of the general term of the linear recursive sequence (3), this opens up possibilities for obtaining other explicit formulas of the entries of the inverse $\mathbb{V}^{-1}$ of the generalized Vandermonde matrices.

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