

Application of the Method of Characteristics in the Analysis of Transient Events in Natural Gas Distribution Networks

Julio César de Almeida¹, Glaúcio Antonio Weigert², Saul Renato da Silva³

¹Department of Mechanical Engineering, Federal University of Paraná - UFPR, Brasil

^{2,3}Department of Operations, Paraná Gas Company - COMPAGAS, Brasil

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Abstract—*In this paper we present the development of a mathematical model for the numerical simulation of one-dimensional compressible transient flows in natural gas distribution networks, based on the method of characteristics. This model was developed to validate transient phenomena in distribution networks of piped natural gas, which is why it was decided to build an experimental network with carbon steel pipe, 140 meters in length and two inches in diameter, in order to simulate situations involving localized ruptures. To achieve the safety conditions and calibration of the instruments, the experiments were initially conducted using compressed air, with subsequent use of natural gas. The study included several cases of leakage in the experimental network, which provided evidence that the results obtained show good agreement with the experimental values, thus justifying the use of the model for real cases.*

I. INTRODUCTION

Mathematics plays an important role in several areas of study — for example, economics, engineering in general, and biological sciences — because through this knowledge it is possible to describe the behavior of some specific systems or phenomena in mathematical terms. Most mathematical formulations for these phenomena lead to rates of change of two or more independent variables, translating these formulations into partial differential equations. The mathematical modeling of these formulations may result in distinct approaches, namely: experimental, analytical, and computational, which can be evaluated jointly or individually. The experimental approach requires a physical model that can facilitate studies involving the analysis of the direct or indirect measurement of the determinant parameters of the evaluated problem, which may, in certain circumstances, be impractical due to the costs and time involved. For the analytical approach, in most cases an adequate solution

cannot be obtained because the mathematical techniques available are not always suitable for determining such solutions. And finally, the computational approach has proven to be an important tool because certain simplifications allow one to obtain a consistent computational model that can be solved using numerical methods.

This work presents a computational mathematical formulation that enables the evaluation of transient events in distribution networks of piped natural gas, by applying the numerical technique based on the method of characteristics, which ensures accurate results with consistent computational times. To validate the said algorithm, we chose to build a corresponding experimental model, through which it became possible to simulate leakage conditions located in a straight section of piping.

II. PARTIAL DIFFERENTIAL EQUATIONS AND THEIR CLASSIFICATION

In engineering, many conventional problems can be described by a partial differential equation (PDE) which can be classified as elliptical, parabolic, or hyperbolic, depending on the category into which the physical phenomenon falls (Lax, 2006). Thus, considering a general PDE in the following form:

$$a \frac{\partial^2 \phi}{\partial x^2} + b \frac{\partial^2 \phi}{\partial x \partial y} + c \frac{\partial^2 \phi}{\partial y^2} + d \frac{\partial \phi}{\partial x} + e \frac{\partial \phi}{\partial y} + \dots - f\phi + g = 0 \tag{1}$$

in which a, b, c, d, e, f, and g may be functions of the independent variables x and y and of the dependent variable ϕ which is defined within a region R of the plane “xy”, can consider that:

$$\Delta = b^2 - 4ac \tag{2}$$

and the PDEs are classified as three distinct types, namely:

if $\Delta < 0$, the equation is said to be elliptic;

if $\Delta = 0$, the equation is said to be parabolic; and

if $\Delta > 0$, the equation is said to be hyperbolic.

In terms of problems involving compressible flows, there are two distinct evaluation approaches: stationary flows and transient flows. In the general stationary problems, the differential equations involved are elliptical, similar to Laplace’s equation. Transient problems include the temporal variation of the magnitudes involved, which is why they are represented by hyperbolic or parabolic differential equations, corresponding to the wave equation and the diffusion equation, respectively. The solution for these systems can be achieved either by numerical methods or by analytical methods — the method of characteristics is both a numerical and analytical method, according to Rodrigues (2010). This method is classically used in the solution of hyperbolic equations, being a good alternative for solving this type of problem, which is why it was chosen for the development of this present work.

Since the PDEs that model physical systems usually have many solutions, it is necessary to impose auxiliary conditions which may characterize a function that represents the solution to the physical problem. Such auxiliary conditions correspond to the boundary conditions and the initial conditions of the problem.

III. MATHEMATICAL FORMULATION FOR THE FLOW OF THE GAS

The transportation of the natural gas along the pipeline can be analyzed as follows (Lurie, 2008): the flow is compressible and transient; the flow is continuous, causing the whole cross-section of the pipe to be filled; the flow can be considered to be one-dimensional (i.e., all the parameters involved depend only on the x coordinate measured along the axis of the pipe and the time t); the cross-sectional area of the pipe can vary along the length; and the piping can be considered to be indestructible, with the interaction between the fluid and the pipe due to vibration problems being negligible.

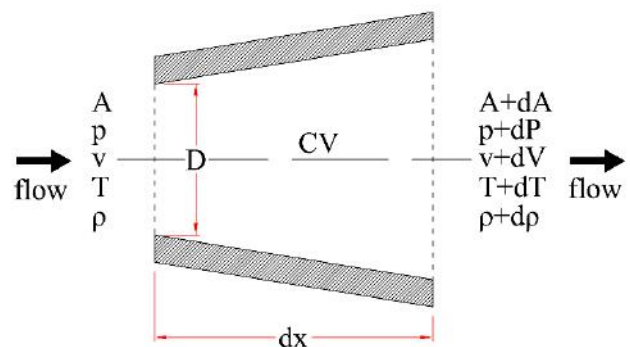
The mathematical models for fluids and gas flows along the pipes are based on physics principles (mechanics and thermodynamics) of the continuum and are obtained from the following fundamental principles: conservation of mass, conservation of momentum, conservation of energy, and a corresponding equation of state.

3.1 – Conservation of mass

The conservation of mass, or continuity equation, in the context of mass flow along a pipe, corresponds to the condition in which the mass of the fluid considered can neither be created nor destroyed. This condition affects the fact that the mass accumulation rate within the control volume (CV), outlined in Figure 1, is equal to the net mass flow in the control surface; that is:

$$\boxed{\text{Mass flow rate in the control volume}} - \boxed{\text{Mass efflux rate in the control volume}} = \boxed{\text{Mass accumulation rate within the control volume}}$$

Fig. 1: Control volume with variable area in a straight pipe section (Almeida et al., 2013)



Mathematically, one can write:

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} + \rho v \frac{1}{A} \frac{\partial A}{\partial x} = 0 \tag{3}$$

given: A = sectional area of the duct;

- v = velocity of the fluid;
- p = pressure of the fluid;
- T = temperature of the fluid;
- ρ = fluid density.

3.2 - Conservation of momentum

The principle of the conservation of momentum corresponds to the application of Newton’s second law of motion to a fluid element. Thus, the net force acting, in the x direction, on the gas within the control volume, corresponds to the algebraic sum of the individual forces present in this system in relation to the same reference volume (Figure 2); that is:

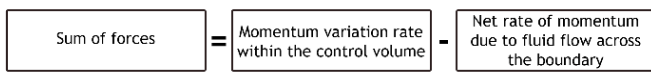
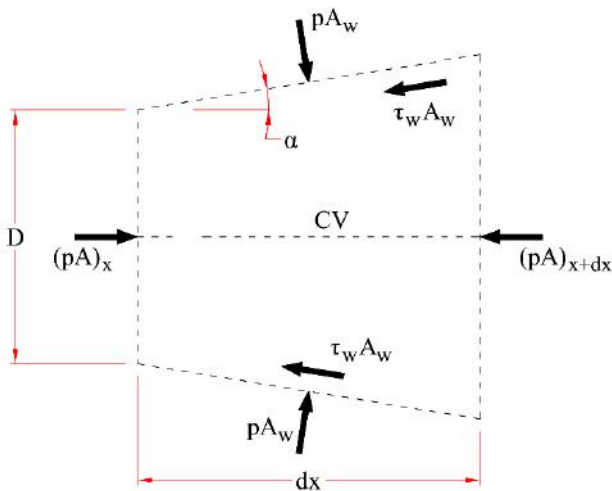


Fig. 2: Forces acting on the control surface (Almeida et al., 2013)



For the present case, due to the symmetry of the control volume it becomes possible to neglect the radial force components, thereby featuring only the presence of forces due to the fluid pressure and shear forces (friction). Electromagnetic and electrostatic forces are negligible and are consequently ignored. Mathematically, this results in:

$$\frac{\partial v}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial x} + v \frac{\partial v}{\partial x} + |v| \frac{2f_D v}{D} = 0 \tag{4}$$

- given: A_w = lateral area of the duct;
- τ_w = shear stress;
- D = diameter of the local duct;

f_D = friction factor.

Designating the last component of this expression as F_a (Darcy friction force), one can also write:

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} + F_a = 0 \tag{5}$$

3.3 Conservation of energy

The equation for conservation of energy is derived from the application of the first Law of Thermodynamics which, when formulated for an open system in terms of rates, corresponds to:

$$\frac{\partial E}{\partial t} = \dot{Q} - \dot{W} + \Delta h \tag{6}$$

that is, the rate of change of the energy within the control volume is equal to the rate of heat transfer into the same (\dot{Q}), minus the work production rate via the control surface (\dot{W}) and, furthermore, added to the net flow of the stagnation enthalpy (Δh).

Since the term corresponding to the work production rate can be considered to be zero due to the walls of the ducts being rigid, one can obtain the conservation of energy equation (explained in terms of the entropy) in the final form:

$$\frac{\partial s}{\partial t} + v \frac{\partial s}{\partial x} = \frac{kR}{c^2} (\dot{q} + F_a v) \tag{7}$$

- given: s = entropy;
- k = ratio of specific heats;
- R = the gas constant;
- c = local speed of sound in the fluid;
- \dot{q} = heat transfer rate per unit mass.

3.4 Differential equations in matrix form

By specific simplifications, with the corresponding application of the equation of state for the ideal gas case, the group of equations (3), (5), and (7) enables us to write the following system of differential equations:

$$\begin{aligned}
 &1 \frac{\partial p}{\partial t} + 0 \frac{\partial v}{\partial t} + 0 \frac{\partial s}{\partial t} + v \frac{\partial p}{\partial x} + c^2 \rho \frac{\partial v}{\partial x} + 0 \frac{\partial s}{\partial x} \dots \\
 &= \rho(k-1)(\dot{q} + vF_a) - \frac{\rho v c^2}{A} \frac{dA}{dx} \\
 &0 \frac{\partial p}{\partial t} + 1 \frac{\partial v}{\partial t} + 0 \frac{\partial s}{\partial t} + \frac{1}{\rho} \frac{\partial p}{\partial x} + v \frac{\partial v}{\partial x} + 0 \frac{\partial s}{\partial x} \dots \\
 &= -F_a \\
 &0 \frac{\partial p}{\partial t} + 0 \frac{\partial v}{\partial t} + 1 \frac{\partial s}{\partial t} + 0 \frac{\partial p}{\partial x} + 0 \frac{\partial v}{\partial x} + v \frac{\partial s}{\partial x} \dots \\
 &= \frac{kR}{c^2} (\dot{q} + vF_a)
 \end{aligned}
 \tag{08}$$

or in the matrix form:

$$\frac{\partial U}{\partial t} + B \frac{\partial U}{\partial x} = f \tag{09}$$

where:

$$U = \begin{pmatrix} p \\ v \\ s \end{pmatrix}$$

$$B = \begin{pmatrix} v & c^2 \rho & 0 \\ \frac{1}{\rho} & v & 0 \\ 0 & 0 & v \end{pmatrix}$$

$$f = \begin{pmatrix} \rho(k-1)(\dot{q} + vF_a) - \frac{\rho v c^2}{A} \frac{dA}{dx} \\ -F_a \\ \frac{kR}{c^2} (\dot{q} + vF_a) \end{pmatrix}$$

$$F_a = f_D \frac{4 |v| v^2}{D v 2}$$

The matrix of coefficients B exhibits real eigenvalues, which are different to each other, thus characterizing a hyperbolic equation set (Godunov, 1994). Among the several methods that can be used to solve hyperbolic problems, finite difference techniques based on the Lax-

Wendroff scheme and the method of characteristics are probably the most used. According to Ames (1965), Lax-Wendroff finite difference methods are adequate for problems that have a well stated solution within the calculation domain. Kruglov et al. (1988), on the other hand, pointed out that in the case of flows that may exhibit abrupt variations in their properties, the method of characteristics can be used to obtain accurate solutions.

The characteristic curves represent the natural coordinate system for solving hyperbolic problems. In this coordinate system, the equations in partial derivatives defined throughout the entire domain can be expressed as ordinary differential equations defined along the characteristic curves. These curves are obtained by solving the equation set, $\det(B - dx/dt I) = 0$, where I is the identity matrix. This equation set expresses the necessary condition for the existence of singularities in the solution of the problem described by Equation (9) (Velásquez et al., 1995). Therefore, the equations of the characteristic curves are:

$$\frac{dx}{dt} = v + c$$

$$\frac{dx}{dt} = v - c$$

$$\frac{dx}{dt} = v$$

The first two characteristic curves in the above equations are called *Mach lines*, while the third one is the trajectory of the fluid particles (path line). The necessary condition for Equation (9) to have a solution along the characteristic curves is expressed by the so-called *compatibility equations*:

Mach line compatibility equations:

$$\begin{aligned}
 (dc)_{mach} \pm \frac{k-1}{2} (dv)_{mach} &= \frac{k-1}{2} \frac{c}{kR} (ds)_{mach} \\
 &+ \frac{k-1}{2} \left[-\frac{vc}{A} \frac{dA}{dx} + (k-1) \frac{1}{c} (\dot{q} + vF_a) \mp F_a \right] dt
 \end{aligned}$$

(10)

Path line compatibility equation:

$$(ds)_{path} = \frac{kR}{c^2} (\dot{q} + vF_a) dt \tag{11}$$

It should be noted that the compatibility equations for the Mach lines in Equation (10) are expressed in terms of the differentials of v , c , and s . However, these equations can be expressed in terms of only two of these differentials by introducing the Riemann variables λ and β , defined according to:

$$\lambda = c + \frac{k-1}{2} v$$

$$\beta = c - \frac{k-1}{2} v$$

By doing so, the following compatibility equations result:

$$(d\lambda)_\lambda = \frac{c}{c_A} (dc_A)_\lambda + \frac{k-1}{2} \left[-\frac{vc}{A} \frac{dA}{dx} + (k-1) \frac{1}{c} (\dot{q} + vF_a) - F_a \right] dt \tag{12}$$

$$(d\beta)_\beta = \frac{c}{c_A} (dc_A)_\beta + \frac{k-1}{2} \left[-\frac{vc}{A} \frac{dA}{dx} + (k-1) \frac{1}{c} (\dot{q} + vF_a) + F_a \right] dt \tag{13}$$

$$(dc_A)_{path} = \frac{k-1}{2} \frac{c_A}{c^2} (\dot{q} + vF_a) dt \tag{14}$$

In Equations (12), (13), and (14), a variable called entropy level (c_A) was introduced instead of entropy s . It is related to the former by the following definition (Benson et al., 1964):

$$c_A = \exp\left(\frac{k-1}{2k} s\right) \tag{15}$$

IV. THE COMPUTATIONAL ROUTINE

The computational model uses two numerical grids — one of them is Eulerian and the other is Lagrangian. The Eulerian grid was built by dividing the pipe length into $(n - 1)$ equal parts, thus identifying n nodes, with two of them at the corresponding duct ends. Furthermore, m points were chosen along the duct, with two of them at the duct ends, thus defining m fluid particles whose positions define the Lagrangian grid.

From initial conditions, the fluid velocity and thermodynamic properties of the fluid are known along the pipe, and starting from these data at (x, t) , the compatibility equations should be integrated along the corresponding characteristic curves, thus giving the solution at $(x, t + \Delta t)$. The time Δt is chosen in order to satisfy the Courant-Fredrich-Lewy stability criterion (Courant et al., 1928), which expresses the necessary condition so that the largest displacement of a perturbation wave does not exceed the distance between neighboring nodes Δx .

V. THE EXPERIMENTAL SETUP

Experimental data were obtained from a pilot network approximately 140 meters long, which was built of carbon-steel piping of 2 inches nominal diameter. In this setup, hypothetical transient events were generated by simulating localized leaks and using both compressed air and natural gas. The experimental apparatus included: five pressure sensors installed along the extension of the pipeline in order to provide records of the pressure variations occurring in the course of each test; a system for regulating pressure at the entry of the experimental network to maintain the pressure of the system within the limits previously established; and, for the test with compressed air, a reservoir (pressure vessel) installed at the beginning of the network. Figure 3 illustrates an isometric schematic of this apparatus, identifying its main components, while Figures 4 and 5 depicts specific details such as the vessel used during the tests, and the main pipeline.

It is worth mentioning that, in order to check the repeatability of the measurements, each test was performed at least three times. Although the opening of the valve was performed manually and without the use of any control device to guarantee the repeatability of this process, all the tests sought to make this opening as quickly as possible. By doing it this way, a delay of about 0.5 s was observed from the moment of activation to the moment that the maximum opening of the valve was reached.

VI. RESULTS AND DISCUSSION

Prior to the comparison of numerical and experimental results, a study was conducted to verify the results obtained after a given number of tests performed in the field under localized leakage conditions. Such tests were initially performed using compressed air as the test fluid and then the gas itself derived from the existing local distribution network.

In Figure 6 it is observed that at the beginning the velocity is zero throughout the whole duct and, after opening the valve located at $x = 137.8$ m, which simulates the rupture that causes the leakage of the air, this velocity increases rapidly at the end of the duct, reaching the sonic condition at the throat section of the valve (Figure 8). In turn, Figure 7 shows that a fall in pressure at the end of the valve occurs simultaneously with the increase in velocity, thus generating a depression wave that propagates in the direction of the tank and reaches it at approximately $t = 0.45$ s. As the depression wave sweeps through the length of the duct, the local pressure decreases and causes the mass of fluid that lies ahead to begin flowing. The depression wave generated at the end of the valve is reflected at the other end as a compression wave which, upon passing through a point, provokes a reduction in the rate at which the local pressure decreases due to the gas leakage. This compression wave reaches the end of the valve at approximately $t = 0.8$ s and returns to being reflected as a depression wave. This process of depression and compression wave propagation causes gas leakages to occur as jets, the intensity of which decreases over time.

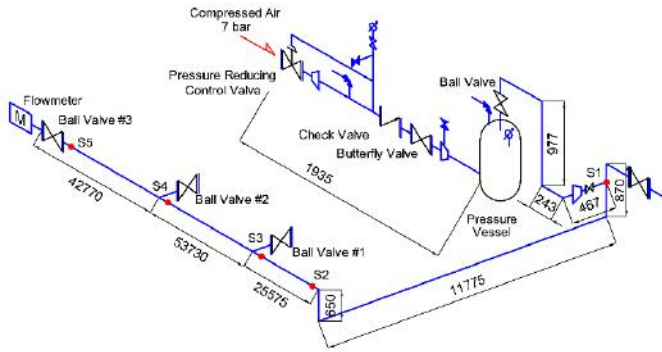


Fig. 3: Schematic of the experimental network



Fig. 4: The pressure vessel



Fig. 5: The main pipeline

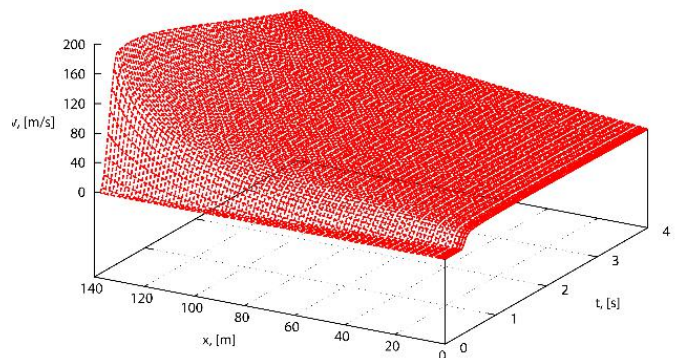


Fig. 6: Velocity of the fluid during emptying of the vessel

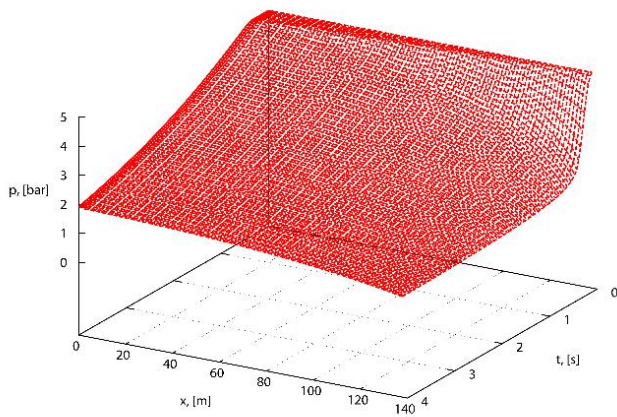


Fig. 7: Pressure of the fluid during emptying of the vessel and pipeline

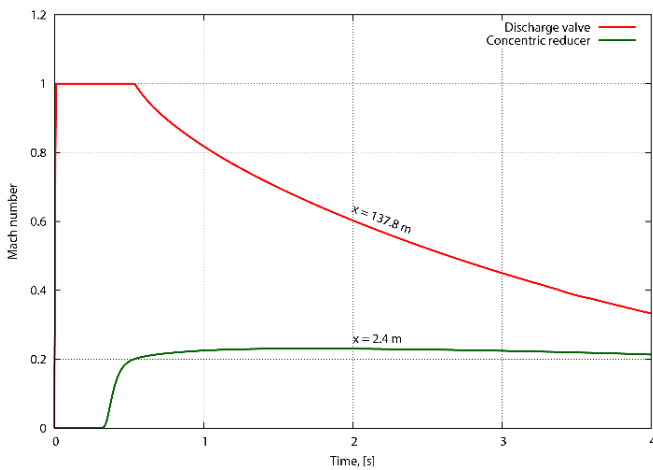


Fig. 8: Mach number at throat sections of the discharge valve and concentric reducer

In Figures 9–10, the experimental data are compared with the results of the simulation for the two cases studied. As can be seen in these graphs, the calculated values show good agreement with the experimental data, despite the final ones displaying noise in the data acquisition system. It should be noted that, despite not using any control device during testing that would guarantee the repeatability of the process of opening the discharge valve, in the computational model the flow area in the valve was considered to vary according to a sinusoidal function and the opening is completed in 0.5 s. It is expected that such an approach would be one of the sources of discrepancies observed between the numerical results and the experimental data.

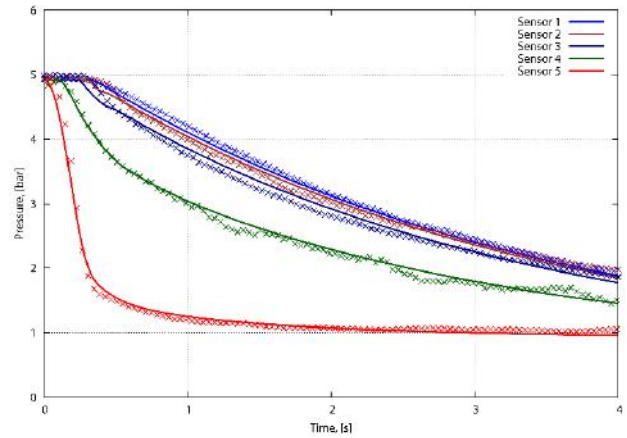


Fig. 9: Numerical and experimental pressure data to a generic test

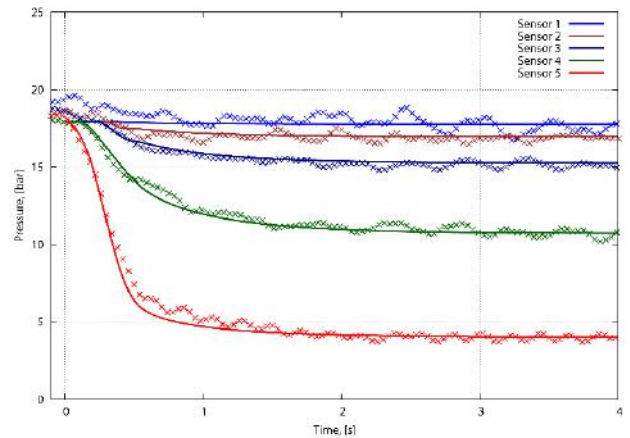


Fig. 10: Numerical and experimental pressure data to a generic test

VII. CONCLUDING REMARKS

A specific mathematical model to describe transient events in natural gas distribution networks, based on the method of characteristics, was presented in this work. Additionally, tests were performed in a pilot pipeline system in order to compare numerical and experimental data. The comparison showed good agreement, thus encouraging the use of the mentioned method for solving problems of this nature.

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