Modified Momentum Euler EquationforWater Wave Modeling

Syawaluddin Hutahaean

Ocean Engineering Program, Faculty of Civil and Environmental Engineering,-Bandung Institute of Technology (ITB), Bandung 40132, Indonesia
syawaluddin1@ocean.itb.ac.id

Abstract*— *In this research, weighted total acceleration for a function $f(x,z,t)$was formulated. This total acceleration equation was done at the Euler momentum equation. Then, the Euler momentum equation was done together with free surface boundary condition equation to formulate water wave constant at the solution of Laplace equation. The velocity potential of the solution of Laplace equation actually consists of two components that were used in this research.

***Keywords*— weighted total acceleration,convective acceleration, complete velocity potential.**

I. INTRODUCTION

Momentum equation is an important basic equation in mathematic modeling of hydrodynamics, including water wave modeling. Momentum equation commonly used in water wave modeling is Euler momentum equation. There is a constraint in this equation, i.e. Euler momentum equation has no hydrodynamic force in the horizontal direction or convective acceleration has a value of zero when velocity potential is substituted to the term. To overcome this problem, weighted total acceleration equation was formulated where there are two weighted coefficients, i.e. at the time$t$ differential term and at the differential term of vertical-$z$direction.

Laplace equation solution consists of two velocity potential components (Dean (1991)). However, only one component that has been used. Equations from water wave constant, i.e. wave number $k$and wave constant G can be formulated using only one velocity potential component, but the value is determined by both the two velocity components. In this research, the water wave surface equation is formulated using the two velocity potential components, then the condition of the water wave surface that has been produced is studied.

II. WEIGHTED TOTAL ACCELERATION

Hutahaean (2019a) formulated weighted total acceleration in a function $f=f(x,t)$, $x$is horizontal axis and t is time, using Taylor series. The formulation of weighted total acceleration in a function$f=f(x,z,t)$,$z$is vertical axis, is done using similar method, therefore the formulation of weighting total acceleration in $f=f(x,z,t)$will be preceded by reviewing the formulation of weighting total acceleration in $f=f(x,t)$to obtain a clearer description.

2.1. Weighted Total Acceleration for the function of$f=f(x,t)$

The changes in the value of a function in a function $f=f(x,t)$for a very smallδxandδtusing Taylor series only until the second derivative is,

$$f(x+\delta x,t+\delta t)=f(x,t)+\delta x\frac{\partial f}{\partial x}+\partial t\frac{\partial f}{\partial t}+\frac{\delta x^2}{2}\frac{\partial^2 f}{\partial x^2}+\delta t\delta x\frac{\partial^2 f}{\partial t\partial x}+\frac{\delta t^2}{2}\frac{\partial^2 f}{\partial t^2}$$

By working on the argument of Courant (1928) that in order to obtain a good result on horizontal velocity $u=\frac{dx}{\gamma dt}$, then weighting coefficient γ, is done which is a positive number, in time differential in Taylor series.

$$f(x+\delta x,t+\gamma\delta t)=f(x,t)+\delta x\frac{\partial f}{\partial x}+\gamma\partial t\frac{\partial f}{\partial t}+\frac{\delta x^2}{2}\frac{\partial^2 f}{\partial x^2}+\gamma\delta t\delta x\frac{\partial^2 f}{\partial t\partial x}+\frac{\gamma^2\delta t^2}{2}\frac{\partial^2 f}{\partial t^2} \quad ....(1)$$

At the limit $\delta x,\delta t$close to zero the following equation is obtained,

$$\frac{Df}{dt}=u\frac{\partial f}{\partial x}+\gamma\frac{\partial f}{\partial t}\text{or}\frac{Df}{dt}=\gamma\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x} \quad ......(2)$$

This equation is weighted total derivative equation or weighted total acceleration for the functionof$f=f(x,t)$where $\gamma$is weighting coefficient.

The method of calculating weighting coeffecient γwill be formulated using Taylor series (1). The second derivative term can be omitted if,

$$\left|\frac{\frac{\delta x^2}{2}\frac{\partial^2 f}{\partial x^2}+\gamma\delta t\delta x\frac{\partial^2 f}{\partial t\partial x}+\frac{\gamma^2\delta t^2}{2}\frac{\partial^2 f}{\partial t^2}}{\delta x\frac{\partial f}{\partial x}+\gamma\partial t\frac{\partial f}{\partial t}}\right|\leq\varepsilon \quad ....(3)$$

https://dx.doi.org/10.22161/ijaers.610.21 ISSN: 2349-6495(P) | 2456-1908(O)

Then it was defined $\delta x = C\delta t = \frac{L}{T}\delta t = \frac{2\pi}{kT}\delta t = \frac{\sigma}{k}\delta t$, where C is wave celerity, k is wave number T is wave period, $\sigma = \frac{2\pi}{T}$ is angular frequency. δx in (3) is substituted with $\frac{\sigma}{k}\delta t$, and the following equation is obtained,

$$\left|\frac{\frac{\sigma^2 \delta t \partial^2 f}{2k^2 \partial x^2} + \frac{\gamma\sigma\delta t}{k}\frac{\partial^2 f}{\partial t \partial x} + \frac{\gamma^2 \delta t}{2}\frac{\partial^2 f}{\partial t^2}}{\frac{\sigma \partial f}{k \partial x} + \gamma\frac{\partial f}{\partial t}}\right| \leq \varepsilon \quad ....(4)$$

The completions of this equation requires a function form of $f = f(x,t)$. And the following sinusoidal function form will be used,

$f(x,t) = \cos kx \cos \sigma t$ ....(5)

This equation is water wave surface equation of the linear wave theory. The derivative of the function is as follows

Table.1: Derivative Equation of (5)

$\frac{\partial f}{\partial x} = -k \sin kx \cos \sigma t$	$\frac{\partial^2 f}{\partial x^2} = -k^2 \cos kx \cos \sigma t$
$\frac{\partial f}{\partial t} = -\sigma \cos kx \sin \sigma t$	$\frac{\partial^2 f}{\partial t \partial x} = k\sigma \sin kx \sin \sigma t$
	$\frac{\partial^2 f}{\partial t^2} = -\sigma^2 \cos kx \cos \sigma t$

Using the condition of $coskx = sinkx = cos\sigma t = sin\sigma t$, the elements of sinusoidal function will cancel out each other as a result of a division. Substitute the derivative equations to (4), the following equation is obtained

$$\left|\frac{\frac{1}{2} - \gamma + \frac{1}{2}\gamma^2}{1+\gamma}\right| \leq \frac{\varepsilon}{\sigma \delta t}$$

The numerator $(1+\gamma)$ is a positive number, then the equation can be written as,

$$\left|\frac{1}{2} - \gamma + \frac{1}{2}\gamma^2\right| \leq \frac{\varepsilon}{\sigma \delta t}(1+\gamma)$$

If equals (=) relation is used, then

$\frac{1}{2} - \gamma + \frac{1}{2}\gamma^2 = \frac{\varepsilon}{\sigma \delta t}(1+\gamma)$ ......(5)

Considering that γ is a positive number, the right side of the equation is a positive number. Therefore, the left side of the equation is also a positive number. The calculation of the value γ can be done by releasing the sign | | in the left side of the equation, i.e. using equation (5).

The calculation of the value γ with (5) requires an input δt. The value of δt, is obtained from the function $f = f(t)$. The approximation of Taylor series for the function is,

$$f(t+\delta t) = f(t) + \delta t \frac{\partial f}{\partial t} + \frac{\delta t^2}{2}\frac{\partial^2 f}{\partial t^2}$$

In order to be able to be used only until the first derivative, then $\left|\frac{\frac{\delta t^2}{2}\frac{\partial^2 f}{\partial t^2}}{\delta t \frac{\partial f}{\partial t}}\right| \leq \varepsilon$ or $\left|\frac{\frac{\delta t}{2}\frac{\partial^2 f}{\partial t^2}}{\frac{\partial f}{\partial t}}\right| \leq \varepsilon$. For the function, $f(t) = \cos \sigma t$; $\frac{\partial f}{\partial t} = -\sigma \sin \sigma t$; $\frac{\partial^2 f}{\partial t^2} = -\sigma^2 \cos \sigma t$ and it is done in a $cos\sigma t = sin\sigma t$ condition, and

$\frac{\frac{\delta t}{2}(-\sigma^2)}{-\sigma} \leq \varepsilon$ or, $\delta t = \frac{2\varepsilon}{\sigma}$ ......(6)

is obtained . Substitution of (6) to (5) obtains

$\gamma = 3$ ...(7)

It is obtained that γ has a constant value, i.e. independent of wave period or the level of accuracy ε.

2.2. Weighted Total Acceleration for the function $f = f(x,z,t)$

To obtain weighted total acceleration equation in a function $f = f(x,z,t)$, the similar method will be done as in the function $f = f(x,t)$, where,

$$f(x+\delta x, z+\gamma_z \delta z, t+\gamma \delta t) = f(x,t) + \delta x \frac{\partial f}{\partial x} + \gamma_z \delta z \frac{\partial f}{\partial z} + \gamma \partial t \frac{\partial f}{\partial t} + \frac{\delta x^2}{2}\frac{\partial^2 f}{\partial x^2} + \gamma_z \delta z \delta x \frac{\partial^2 f}{\partial z \partial x} + \frac{(\gamma_z \delta z)^2}{2}\frac{\partial^2 f}{\partial z^2} + \gamma \delta t \delta x \frac{\partial^2 f}{\partial t \partial x} + \gamma \gamma_z \delta t \delta z \frac{\partial^2 f}{\partial t \partial z} + \frac{(\gamma \partial t)^2}{2}\frac{\partial^2 f}{\partial t^2} \quad ... .(8)$$

In (8), for $\delta z = \delta x$, it is meant that $\gamma_z \delta z \frac{\partial f}{\partial z} = \delta z\left(\gamma_z \frac{\partial f}{\partial z}\right)$, therefore in a change of z for $\delta z = \delta x$, the value of the first derivative function against z is $\left(\gamma_z \frac{\partial f}{\partial z}\right)$, and so also $\gamma_z \delta z \delta x \frac{\partial^2 f}{\partial z \partial x}$ is meant $\delta z \delta x \left(\gamma_z \frac{\partial^2 f}{\partial z \partial x}\right)$ and $\frac{(\gamma_z \delta z)^2}{2}\frac{\partial^2 f}{\partial z^2}$ which means as $\frac{\delta z^2}{2}\left(\gamma_z^2 \frac{\partial^2 f}{\partial z^2}\right)$. As in the previous section, the value of δx and δz is $\delta x = \delta z = C\delta t = \frac{L}{T}\delta t = \frac{2\pi \delta t}{kT} = \frac{\sigma \delta t}{k}$. Then, a function $f(x,z,t)$ is reviewed with the following form.

$f(x,z,t) = coskxcoshk(h+z)\ cos\sigma t$....(9)

At $z = 0$, $c_1 = cosh(kh)$ and $c_2 = sinh(kh)$ are defined and done in the deep water where $tankh = 1$ with the value of $kh = 2.0\pi$. Then $c_1 = cosh(2.0\pi) = c_2 = \sinh(2\pi)$, and (8) is done in a condition of $coskx = sinkx = cos\sigma t = sin\sigma t$, then the sinusoidal function cancelled out each other. The derivative equations (9) can be written in the forms shown in Table (2).

Table.2: Differential of (9).

$\frac{\partial f}{\partial x} = -kc_1$	$\frac{\partial^2 f}{\partial x^2} = -k^2 c_1$	$\frac{\partial^2 f}{\partial t \partial x} = \sigma k c_1$

https://dx.doi.org/10.22161/ijaers.610.21 ISSN: 2349-6495(P) | 2456-1908(O)

$\frac{\partial f}{\partial z} = kc_2$	$\frac{\partial^2 f}{\partial x \partial z} = -k^2 c_2$	$\frac{\partial^2 f}{\partial t \partial z} = -\sigma k c_2$
$\frac{\partial f}{\partial t} = -\sigma c_1$	$\frac{\partial^2 f}{\partial z^2} = k^2 c_1$	$\frac{\partial^2 f}{\partial t^2} = -\sigma^2 c_1$

To simplify the writing, the followings are defined

$$A = \frac{\delta x^2}{2}\frac{\partial^2 f}{\partial x^2} + \gamma_z \delta z \delta x \frac{\partial^2 f}{\partial z \partial x} + \frac{(\gamma_z \delta z)^2}{2}\frac{\partial^2 f}{\partial z^2}$$

$$+\gamma \delta t \delta x \frac{\partial^2 f}{\partial t \partial x} + \gamma \gamma_z \delta t \delta z \frac{\partial^2 f}{\partial t \partial z} + \frac{(\gamma \partial t)^2}{2}\frac{\partial^2 f}{\partial t^2}$$

$$B = \delta x \frac{\partial f}{\partial x} + \gamma_z \delta z \frac{\partial f}{\partial z} + \gamma \partial t \frac{\partial f}{\partial t}$$

In order for (8) to be able to be used with only the first derivate, then

$\left|\frac{A}{B}\right| \le \varepsilon$ ....(9)

The substitution of differential equations in Table (2) to (9) will obtain,

$$-\frac{\gamma^2}{2}c_1 + \gamma - \frac{c_1}{2} - \frac{\varepsilon}{\sigma \delta t}(-\gamma c_1 - c_1)$$

$$-\left(\gamma + 1 + \frac{\varepsilon}{\sigma \delta t}\right) c_2 \gamma_z + \frac{c_1}{2}\gamma_z^2 = 0$$

Substitute δt from (6), $\delta t = \frac{2\varepsilon}{\sigma}$

$$-\frac{\gamma^2}{2}c_1 + \gamma - \frac{c_1}{2} - \frac{1}{2}(-\gamma c_1 - c_1)$$

$-\left(\gamma + 1 + \frac{1}{2}\right) c_2 \gamma_z + \frac{c_1}{2}\gamma_z^2 = 0$ ....(10)

With (10), γ_z can be calculated where γ is a known from(7). With an input $\gamma = 3$, $\gamma_z = 1{,}630$ is obtained for $c_1 = \cosh(2.0\pi)$ and $c_2 = \sinh(2.0\pi)$ where $c_1 = c_2$.

As a result the second derivative in (8) can be omitted and the total derivative equation for function $f = f(x,z,t)$ is

$\frac{Df}{dt} = \gamma \frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} + \gamma_z w \frac{\partial f}{\partial z}$ ......(11)

III. A Complete Velocity Potential Equation

By completing the Laplace equation with separation variable method and after doing the time periodic boundary condition and lateral periodic boundary condition, Dean (1991) obtained velocity potential equation that consisted of two potential velocities, i.e.

$$\varphi(x,z,t) = A \cos k\, x (Ce^{kz} + De^{-kz}) \sin \sigma\, t$$

$+B \sin k\, x (Ce^{kz} + De^{-kz}) \sin \sigma\, t$...(12)

$\varphi(x,z,t) = \varphi_A(x,z,t) + \varphi_B(x,z,t)$...(13)

$\varphi_A(x,z,t) = A \cos k\, x (Ce^{kz} + De^{-kz}) \sin \sigma\, t$ ...(14)

$\varphi_B(x,z,t) = B \sin k\, x (Ce^{kz} + De^{-kz}) \sin \sigma\, t$ ...(15)

There are four constants that should be determined, i.e. A, B, C and D. Hutahaean (2019b) has shown that the two equations have similar constant value, or in other words there is only one constant value in velocity potential total (12). However, in the next section it will be proven again with another method that (12) has one constant value.

Equation (12) can be written as,

$\varphi(x,z,t) = (A \cos k\, x + B \sin k\, x)$ $(Ce^{kz} + De^{-kz}) \sin \sigma\, t$ ....(13)

At a condition of $\cos kx = \sin kx$, (13) can be written as

$\varphi(x,z,t) = (A + B) \cos k\, x$ $(Ce^{kz} + De^{-kz}) \sin \sigma\, t$.....(14)

or

$\varphi(x,z,t) = (A + B) \sin k\, x$ $(Ce^{kz} + De^{-kz}) \sin \sigma\, t$ ....(15)

The constants of A, B, C and D will be formulated using (14) and (15), where it will be proven that either using (14) or (15) similar constant will be obtained. The formulation is done by doing kinematic bottom boundary condition on flat bottom, as was done by Dean (1991).

a. Alternative I

The constants A, B, C and D will be determined using (14) where water particle velocity at the vertical-z direction is

$$w = -\frac{\partial \varphi}{\partial z} = -(A+B) k \cos k\, x$$

$$(Ce^{kz} - De^{-kz}) \sin \sigma\, t$$

Substitute equation for w to the kinematic bottom boundary condition equation $w_{-h} = -u_{-h}\frac{\partial h}{\partial x}$, where at flat bottom $\frac{\partial h}{\partial x} = 0$,

$$-(A+B) k \cos k\, x (Ce^{-kh} - De^{kh}) \sin \sigma\, t = 0$$

The equation is divided by $-(A+B)k \cos kx \sin \sigma t$ for $\cos kx \neq 0$ and $\sin \sigma t \neq 0$

$Ce^{-kh} - De^{kh} = 0$ or $C = De^{2kh}$. Substitute C (14)

$$\Phi(x,z,t) = (A+B) \cos k\, x$$

$$(De^{2kh} e^{kz} + De^{-kz}) \sin \sigma\, t$$

or

$$\Phi(x,z,t) = (A+B) De^{kh} \cos k\, x$$

$$\left(e^{k(h+z)} + e^{-k(h+z)}\right) \sin \sigma\, t$$

A new constant is defined

$G_A = (A+B) De^{kh}$ .....(16)

$\Phi(x,z,t) = G_A \cos k\, x \left(e^{k(h+z)} + e^{-k(h+z)}\right) \sin \sigma\, t$ ....(17)

a. Alternative II

The constants A, B, C and D will be determined using (15),

$$w = -\frac{\partial \varphi}{\partial z} = -(A+B) k \sin k\, x$$

$$(Ce^{kz} - De^{-kz}) \sin \sigma\, t$$

Substitute equations for u and w to the kinematic bottom boundary condition equation

$w_{-h} = -u_{-h}\frac{\partial h}{\partial x}$, where $\frac{\partial h}{\partial x} = 0$

https://dx.doi.org/10.22161/ijaers.610.21 ISSN: 2349-6495(P) | 2456-1908(O)

$-(A+B)k\,sin\,k\,x(Ce^{-kh} - De^{kh})\,sin\,\sigma\,t = 0$

The equation is divided by $-(A+B)ksinkxsin\sigma t$ for $sinkx \neq 0$ and $sin\sigma t \neq 0$

$Ce^{-kh} - De^{kh} = 0$ or $C = De^{2kh}$. Substitute C to (15)

$$\Phi(x,z,t) = (A+B)\,sin\,k\,x\,(De^{2kh}e^{kz} + De^{-kz})\,sin\,\sigma\,t$$

or

$$\Phi(x,z,t) = (A+B)De^{kh}\,sin\,k\,x\big(e^{k(h+z)} + e^{-k(h+z)}\big)\,sin\,\sigma\,t$$

A new constant is defined

$G_B = (A+B)De^{kh}$ .....(18)

$$\Phi(x,z,t) = G_B\,cos\,k\,x\,\big(e^{k(h+z)} + e^{-k(h+z)}\big)\,sin\,\sigma\,t \text{ ....19)}$$

From(16) and (18) obtained that $G_A = G_B = G$, so it is proven that in (1) there is only one wave constant value G, then (7) becomes

$$\Phi(x,z,t) = G(cos\,k\,x + sinkx)\big(e^{k(h+z)} + e^{-k(h+z)}\big)\,sin\,\sigma\,t \text{....(20)}$$

The hyperbolic function equation is, $e^{k(h+z)} + e^{-k(h+z)} = 2coshk(h+z)$, (13) becomes

$\Phi(x,z,t) = 2G(cos\,kx + sinkx)coshk(h+z)\,sin\,\sigma\,t$

Defined $G = 2G$

$$\Phi(x,z,t) = G(cos\,kx + sinkx)coshk(h+z)\,sin\,\sigma\,t \text{ ....(21)}$$

A complete velocity potential equation is obtained with the form as in (21). In that equation, there are still two wave constants where the form should be known, i.e. wave number k and wave constant G. Considering that the values of wave number k and wave constant G is similar along the wave curve, then the calculation of the two parameters will be done at the point of characteristic where $coskx = sinkx$, at this condition,(21) becomes,

$\Phi(x,z,t) = 2Gcoskxcoshk(h+z)\,sin\,\sigma\,t$....(26)

The particle velocity in horizontal-$x$direction is,

$u = -\frac{\partial\Phi}{\partial x} = 2Gksinkxcoshk(h+z)sin\sigma t$ ....(27)

The particle velocity in vertical-$z$direction is,

$w = -\frac{\partial\Phi}{\partial z} = -2Gkcos\,kx\,\sinh k(h+z)\,\sin\sigma t$ .....(28)

IV. Application of Weighted Total Acceleration on Euler Momentum Equation

From (28), the total derivative for horizontal $x$direction velocity is,

$$\frac{Du}{dt} = \gamma\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \gamma_z w\frac{\partial u}{\partial z}$$

With this total derivative equation, the Euler momentum equation in horizontal-$x$direction becomes,

$$\gamma\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \gamma_z w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

By doing the characteristic of irrotational flow, $\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$ obtained,

$\gamma\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(u^2 + \gamma_z w^2) = -\frac{1}{\rho}\frac{\partial p}{\partial x}$ ....(29)

Total derivative equation for vertical velocity in axis-z direction.

$$\frac{Dw}{dt} = \gamma\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + \gamma_z w\frac{\partial w}{\partial z}$$

The Euler momentum equation in vertical-$z$direction becomes,

$$\gamma\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + \gamma_z w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

The execution of irrotational flow characteristic, $\frac{\partial w}{\partial x} = \frac{\partial u}{\partial z}$

$\gamma\frac{\partial w}{\partial t} + \frac{1}{2}\frac{\partial}{\partial z}(u^2 + \gamma_z w^2) = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$.....(30)

(29) and (30) are modified Euler momentum equations, where there are time weighting coefficient γ and weighting coefficient vertical $z$direction of weighting coefficient, i.e.γ_z. Using (30) pressure p equation will be formulated where (30) is written as an equation for pressure p.

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = \gamma\frac{\partial w}{\partial t} + \frac{1}{2}\frac{\partial}{\partial z}(u^2 + \gamma_z w^2) + g$$

This equation is multiplied by dz and integrated against vertical-z axis.

$$\frac{p}{\rho} = \gamma\int_z^\eta \frac{\partial w}{\partial t}dz + \frac{1}{2}(u_\eta^2 + \gamma_z w_\eta^2) - \frac{1}{2}(u^2 + \gamma_z w^2) + g(\eta - z)$$

Differentiated against horizontal-$x$axis

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \gamma\frac{\partial}{\partial x}\int_z^\eta \frac{\partial w}{\partial t}dz + \frac{1}{2}\frac{\partial}{\partial x}(u_\eta^2 + \gamma_z w_\eta^2) - \frac{1}{2}\frac{\partial}{\partial x}(u^2 + \gamma_z w^2) + g\frac{\partial\eta}{\partial x}$$

Substituted to (29)

$$\gamma\frac{\partial u}{\partial t} + \gamma\frac{\partial}{\partial x}\int_z^\eta \frac{\partial w}{\partial t}dz + \frac{1}{2}\frac{\partial}{\partial x}(u_\eta^2 + \gamma_z w_\eta^2) = -g\frac{\partial\eta}{\partial x} \text{ ....(31)}$$

The completion of $\frac{\partial}{\partial x}\int_z^\eta \frac{\partial w}{\partial t}dz$ is done using velocity potential (21), where the particle velocity in horizontal direction is in equation (27), and the particle velocity in vertical-$z$direction (28). From (28) the following is obtained,

$$\frac{\partial w}{\partial t} = -2Gk\sigma coskxsinhk(h+z)cos\sigma t$$

This equation is integrated against time t,

$$\int_z^\eta \frac{\partial w}{\partial t}dz = -2G\sigma coskx\big(coshk(h+\eta) - coshk(h+z)\big)cos\sigma t$$

Then, it is differentiated against horizontal-x axis

$$\frac{\partial}{\partial x}\int_z^\eta \frac{\partial w}{\partial t}dz = Gk\sigma sinkx$$
$$\left(coshk(h+\eta) - coshk(h+z)\right)cos\sigma t$$

Equation (27) is differentiated against time t, $\frac{\partial u}{\partial t} = 2Gk\sigma sinkxcoshk(h+z)cos\sigma t$, and it is seen that this form is in $\frac{\partial}{\partial x}\int_z^\eta \frac{\partial w}{\partial t}dz$, so the following relation is obtained

$$\frac{\partial}{\partial x}\int_z^\eta \frac{\partial w}{\partial t}dz = \left(\frac{\partial u_\eta}{\partial t} - \frac{\partial u}{\partial t}\right)$$

Substitute this equation to (31)

$\gamma\frac{\partial u_\eta}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left(u_\eta^2 + \gamma_z w_\eta^2\right) = -g\frac{\partial \eta}{\partial x}$.....(32)

This equation is a surface momentum equation that will be used in the calculation of $G$and$k$.

V. THE FORMULATION OF AN EQUATION FOR THE CALCULATION OF G AND k

As has been mentioned in the previous section that the calculation of $G$and $k$is done in the point of characteristic where $coskx = sinkx$. Therefore, (27) is used as the particle velocity in horizontal $x$direction and (28) is particle velocity equation in vertical z direction.

5.1 Wave number conservation equation

In the formulation of an equation for the calculation of $G$and$k$in the following sub-chapter, the wave number conservation equation will be done. The equation come from the principle of variable separation at the completion of Laplace equation, i.e. that velocity potential is considered as a multiplication of three functions, i.e. $\Phi(x,z,t) = X(x)Z(z)T(t)$where$X(x)$is just a function-$x$, $Z(z)$is just a function-$z$and$T(t)$ is just a function-t. In this case $Z(z) = coshk(h+z)$. As just function-$z$then, $\frac{\partial Z(z)}{\partial x} = 0$.

$$\frac{\partial coshk(h+z)}{\partial x} = sinhk(h+z)\frac{\partial k(h+z)}{\partial x} = 0$$

For $sinhk(h+z)$is not equal to zero, then

$\frac{\partial k(h+z)}{\partial x} = 0$ ....(33)

This equation (33) is called wave number conservation equation. This means that all area of calculation has similar values for the function $tanhk(h+z)$, $coshk(h+z)$ dan $sinhk(h+z)$. As deep water, it can be defined as water depth where$tanhk(h+\eta) = 1$, where$\eta = \eta(x.t)$is the water surface elevation against still water level. Bearing in mind that the wave number conservation equation or law, in the entire domain applies

$tanhk(h+\eta) = 1$ ....(34)

In this research, the following is used

$k(h+\eta) = 2.0\,\pi$ ....(35)

Where$tanh(2.0\pi) = 0.999993$.

$sinhk(h+\eta) = coshk(h+\eta) = sinh(2.0\pi) = cosh(2.0\pi)$ ......(36)

Bearing in mind this wave number conservation law, then even though the weighting coefficient is formulated in deep water condition, it will also apply in other depths.

5.2. Substitute Velocity Potential to Momentum Equation

From (27),

$$u = 2Gksinkxcoshk(h+z)sin\sigma t$$
$$\frac{\partial u}{\partial x} = 2Gk^2coskxcoshk(h+z)sin\sigma t$$
$$u\frac{\partial u}{\partial x} = 4G^2k^3sinkxcoskxcosh^2k(h+z)sin^2\sigma t$$

At $z = \eta$

$u\frac{\partial u}{\partial x} = 4G^2k^3sinkxcoskxcosh^2k(h+\eta)sin^2\sigma t$.....(37)

From (28)

$$w = -2Gkcos\,kx\sinh k(h+z)\sin\sigma t$$
$$\frac{\partial w}{\partial x} = 2Gk^2sin\,kx\sinh k(h+z)\sin\sigma t$$
$$w\frac{\partial w}{\partial x} = -4G^2k^3sinkxcoskxsinh^2k(h+z)sin^2\sigma t$$

At $z = \eta$

$w\frac{\partial w}{\partial x} = -4G^2k^3sinkxcoskxsinh^2k(h+\eta)sin^2\sigma t)$ ..(38)

In deep water where$tanhk(h+\eta) = 1$, then$sinhk(h+\eta) = coshk(h+\eta)$. Substitute (37) and (38) to the convective velocity,

$$\left(u\frac{\partial u}{\partial x} + \gamma_z w\frac{\partial w}{\partial x}\right)_{z=\eta} =$$
$$(1-\gamma_z)G^2k^3sin\,kxcoskx\cosh^2 k(h+\eta)\sin^2\sigma t$$

At the characteristic point, i.e. a point where $coskx = sinkx = cos\sigma t = sin\sigma t$, where$\eta = \frac{A}{2}$,

$$\left(u\frac{\partial u}{\partial x} + w\gamma_z\frac{\partial w}{\partial x}\right)_{z=\eta} =$$

$\frac{1}{4}(1-\gamma_z)G^2k^3\cosh^2 k\left(h+\frac{A}{2}\right)$ .....(39)

If in (39) $\gamma_z = 1$ is used, then $\left(u\frac{\partial u}{\partial x} + w\gamma_z\frac{\partial w}{\partial x}\right)_{z=\eta} = 0$ will be obtained. So, it is found that if at the term $w\frac{\partial w}{\partial x}and\gamma_z = 1$is used or without weighting coefficient γ_z, hydroynamic force of the surface in horizontal direction has a value of zero or there is no hydrodynamicforce. From (27), at the characteristic point,

$\frac{\partial u}{\partial t} = Gk\sigma\cosh k\left(h+\frac{A}{2}\right)$ .....(40)

Substitute (39) and (40) to (32),

$$\gamma\sigma Gk\cosh k\left(h+\frac{A}{2}\right)$$
$+\frac{1}{2}(1-\gamma_z)G^2k^3\cosh^2 k\left(h+\frac{A}{2}\right) = -g\frac{\partial \eta}{\partial x}$....(41)

Where $g\frac{\partial \eta}{\partial x}$is worked at the characteristic point. This equation is a relation between $G$and$k$where wave amplitude $A$is the input.

https://dx.doi.org/10.22161/ijaers.610.21 ISSN: 2349-6495(P) | 2456-1908(O)

5.2 The formulation of wave amplitude function

The weighted total acceleration equation (2), done at the water wave surface equation $\eta = \eta(x,t)$, obtained $\frac{D\eta}{dt} = \gamma\frac{\partial\eta}{\partial t} + u_\eta\frac{\partial\eta}{\partial x}$. The original kinematic free surface boundary condition (KFSBC) equation is, $w_\eta = \frac{\partial\eta}{\partial t} + u_\eta\frac{\partial\eta}{\partial x}$. By comparing the two equations, then the KFSBC equation should be in the form of $w_\eta = \gamma\frac{\partial\eta}{\partial t} + u_\eta\frac{\partial\eta}{\partial x}$, or

$$\gamma\frac{\partial\eta}{\partial t} = w_\eta - u_\eta\frac{\partial\eta}{\partial x} \quad ...(42)$$

Substitute u from (27) and w from (28) and done at $z = \eta$,

$$\gamma\frac{\partial\eta}{\partial t} = -2Gk sinhk(h+\eta)coskxsin\sigma t$$
$$-2Gk coshk(h+\eta)sinkxsin\sigma t\ \frac{\partial\eta}{\partial x} \quad ......(43)$$

Water wave surface equation was obtained by integrating (43) against time t. The right side of the equation is a non-linear function against time t of which the integration completion is difficult. However, there is an argument that can simplify the integration (43) completion. First bearing in mind (36), i.e. $coshk(h+\eta) = \cosh(2.0\pi) = constant$. Then, (43) is written as,

$$\gamma\frac{\partial\eta}{\partial t} = -2Gk$$
$$\left(coskxsinhk(h+\eta) + sinkxcoshk(h+\eta)\frac{\partial\eta}{\partial x}\right)sin\sigma t \quad ...(44)$$

In (44) the one that is the function of time t is only the element of $sin\sigma t$. In addition, as a periodical function against time t, the element $-2Gk\left(coskxsinhk(h+\eta) + 2sinkxcoshk(h+\eta)\frac{\partial\eta}{\partial x}\right)$ should be a constant number against time t. Thus, the integration (44) against time t, is sufficient by integrating only the $sin\sigma t$ element, obtained

$$\eta(x,t) = \frac{2Gk}{\gamma\sigma}$$
$$\left(coskxsinhk(h+\eta) + sinkxcoshk(h+\eta)\frac{\partial\eta}{\partial x}\right)cos\sigma t \quad ....(45)$$

At the characteristics point, (45) can be written as

$$\eta(x,t) = \frac{2Gk}{\gamma\sigma}$$
$$\left(sinhk(h+\eta) + coshk(h+\eta)\frac{\partial\eta}{\partial x}\right)coskxcos\sigma t \quad ......(46)$$

The form $coskx$ was selected becauseit has been determined that the velocity potential component that was used is $coskx$ componentIt is defined

$$A = \frac{2Gk}{\gamma\sigma}\left(sinhk(h+\eta) + coshk(h+\eta)\frac{\partial\eta}{\partial x}\right)$$

Then (46) becomes

$$\eta(x,t) = Acoskxcos\sigma t \quad ....(47)$$

At the characteristic point, then $\eta = \frac{A}{2}$, wave amplitude function equation,

$$A = \frac{2Gk}{\gamma\sigma}\left(sinhk\left(h+\frac{A}{2}\right) - coshk\left(h+\frac{A}{2}\right)\frac{kA}{2}\right)$$

From (36) where $sinhk\left(h+\frac{A}{2}\right) = coshk\left(h+\frac{A}{2}\right)$ the wave amplitude function equation becomes,

$$A = \frac{2Gk}{\gamma\sigma}coshk\left(h+\frac{A}{2}\right)\left(1-\frac{kA}{2}\right) ....(48)$$

5.3 Equation for the calculation of k and G

Substitute (47) to (41) at the characteristic point

$$\gamma\sigma Gk\cosh k\left(h+\frac{A}{2}\right) + \frac{1}{2}(1-\gamma_z)G^2k^3\cosh^2 k\left(h+\frac{A}{2}\right) = g\frac{kA}{2}$$

Substitute wave amplitude function,

$$G\gamma\sigma\cosh k\left(h+\frac{A}{2}\right) + \frac{1}{2}(1-\gamma_z)G^2k^2\cosh^2 k\left(h+\frac{A}{2}\right) = g\frac{Gk}{\gamma\sigma}\cosh k\left(h+\frac{A}{2}\right)\left(1-\frac{kA}{2}\right)$$

The equation is divided by $\frac{Gk}{\gamma\sigma}\cosh k\left(h+\frac{A}{2}\right)$,

$$\gamma^2\sigma^2 + \frac{\gamma\sigma}{2}(1-\gamma_z)Gk^2\cosh k\left(h+\frac{A}{2}\right) = gk\left(1-\frac{kA}{2}\right)$$

Wave amplitude equation is written as an equation for G, i.e.

$$G = \frac{A\gamma\sigma}{2k\cosh k\left(h+\frac{A}{2}\right)\left(1-\frac{kA}{2}\right)} \quad ....(49)$$

and substitute it to the last equation,

$$\gamma^2\sigma^2\left(1-\frac{kA}{2}\right) + \frac{\gamma^2\sigma^2}{4}(1-\gamma_z)kA = gk\left(1-\frac{kA}{2}\right)^2 \quad .....(50)$$

The calculation of the value k with this equation using Newton-Rhapson method requires initial estimation of k for the initial value of the iteration. The initial value of k can be obtainedby ignoring convective acceleration, then (50) becomes

$$\gamma^2\sigma^2 = gk\left(1-\frac{kA}{2}\right) \quad ....(51)$$

This equation is the quadratic equation of wave number k that can be easily completed. The use of (51) maximum value of wave amplitude A in a wave period in deep water is obtained, i.e. if the determinant D from (51) has a value of zero.

$$A_{max} = \frac{g}{2\gamma^2\sigma^2} \quad ....(52)$$

The value of G can be calculated using (49).

VI. THE FORMULATION OF WATER WAVE SURFACE EQUATION.

Water wave surface equation is formulated using a complete velocity potential equation, i.e. equation (21).

By using (21), particle velocity in horizontal-$x$direction and particle velocity in vertical-$z$direction are consecutively,

$$u = Gk(\sin kx - \cos kx)\cosh k(h+z)\sin\sigma t$$

$$w = -Gk(\cos kx + \sin kx)\sinh k(h+z)\sin\sigma t$$

The two particle velocity equations are done at $z = \eta$and substituted to equation KFSBC (42),

$$\gamma\frac{\partial\eta}{\partial t} = -Gk(\cos kx + \sin kx)\sinh k(h+\eta)\sin\sigma t - Gk(\sin kx - \cos kx)\cosh k(h+\eta)\sin\sigma t\frac{\partial\eta}{\partial x}$$

....(53)

As in the previous section, the water wave surface equation is obtained by integrating (53) against time$t$, where the integration is sufficient to be done only at the $\sin\sigma t$element,

$$\eta(x,t) = \frac{Gk}{\gamma\sigma}(\cos kx + \sin kx)\sinh k(h+\eta)\cos\sigma t + \frac{Gk}{\gamma\sigma}(\sin kx - \cos kx)\cosh k(h+\eta)\cos\sigma t\frac{\partial\eta}{\partial x}$$

In the deep water the equation can be written as,

$$\eta(x,t) = c_0\left((c_2 + c_1) + (c_1 - c_2)\frac{\partial\eta}{\partial x}\right)c_3 \text{......(54)}$$

where,to simplify the writing $c_0 = \frac{Gk}{\gamma\sigma}\cosh k(h+\eta)$, $c_1 = \sin kx$, $c_2 = \cos kx$ dan $c_3 = \cos\sigma t$are defined. Equation (54) is differentiated against horizontal-x axis

$$\frac{\partial\eta}{\partial x} = c_0 k\left((-c_1 + c_2) + (c_2 + c_1)\frac{\partial\eta}{\partial x}\right)c_3 \text{ ....(55)}$$

Equation (54) is water wave surface equation that is used to calculate water surface elevation where $\frac{\partial\eta}{\partial x}$in (54) is calculated using (55). ηin $c_0 = \frac{Gk}{\gamma\sigma}\cosh k(h+\eta)$is calculated using the equation,

$$\eta(x,t) = A(\cos kx + \sin kx)\cos\sigma t \text{ ....(56)}$$

Whereas $\frac{\partial\eta}{\partial x}$in (55) it is calculated with,

$$\frac{\partial\eta}{\partial x} = Ak(-\sin kx + \cos kx)\cos\sigma t \text{ ....(57)}$$

VII. THE RESULTS OF THE EQUATION.

7.1 The characteristic of water wave surface.

In the calculations that will be done in this section, the value of $\gamma = 3.0$and$\gamma_z = 1.630$are used and the calculation is done in the deep water. Deep water depth $h_0$is obtained with the following equation

$$h_0 = \frac{1}{k}\left(2.0\pi - \frac{kA}{2}\right) \text{ ....(58)}$$

Where$A$is calculated using (52).

Table.3: The result of calculation of wave parameter and other characteristic

T (sec)	H (m)	L (m)	$\frac{H}{L}$	$\frac{H}{A}$	$\frac{\eta_{max}}{H}$
6	1,409	5,026	0,28	2,865	0,851
7	1,918	6,842	0,28	2,865	0,851
8	2,506	8,936	0,28	2,865	0,851
9	3,171	11,309	0,28	2,865	0,851
10	3,915	13,962	0,28	2,865	0,851
11	4,737	16,894	0,28	2,865	0,851
12	5,638	20,105	0,28	2,865	0,851
13	6,617	23,595	0,28	2,865	0,851
14	7,674	27,365	0,28	2,865	0,851
15	8,81	31,413	0,28	2,865	0,851

Using water wave surface equation, the elevation of wave crest $\eta_{max}$and the elevation of wave trough $\eta_{min}$are calculated. The wave height is $H = \eta_{max} - \eta_{min}$, whereas Wilson (1963) criteria is$\frac{\eta_{max}}{H}$.Table (3) presented the result of the calculations of wave height, wavelength, wave steepness, and the comparison of wave height $H$and wave amplitude A.

Wave-steepness $\frac{H}{L} = 0.280$, where considering the calculation used maximum wave Amplitude A that was calculated using (52), then wave steepness is critical wave steepness.

Table.4: Types of wave, according to Wilson criteria (1963)

Wave Type	$\frac{\eta_{max}}{H}$
Airy waves	< 0.505
Stoke's waves	< 635
Cnoidal waves	$0.635 < \frac{\eta_{max}}{H} < 1$
Solitary waves	$= 1$

The critical wave steepness is bigger than the criteria ofMichell (1893) i.e.$\frac{H}{L} = 0.142$. The comparison between wave height and wave amplitude is$\frac{H}{A} = 2.865$which is bigger than 2. Therefore, therelation between wave height and wave amplitude is$H = 2A$ cannot be used. The obtained Wilson parameter is $\frac{\eta_{max}}{H} = 0.851$. Based on Wilson criteria (1963), Table (4), the value of the parameter shows that the wave profile has a cnoidal wave type, with wave profile presented inFig.1.and Fig.2.for wave period $T = 8$ sec.

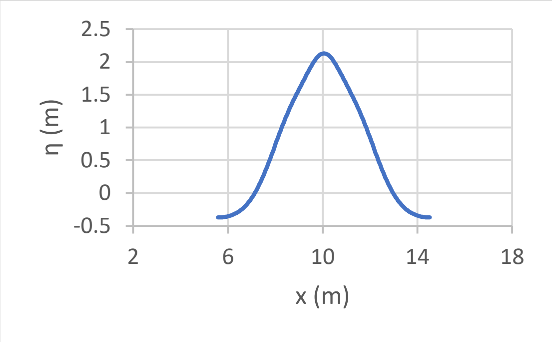

Fig.1. Wave profile with wave period of 8 sec., in one wave length

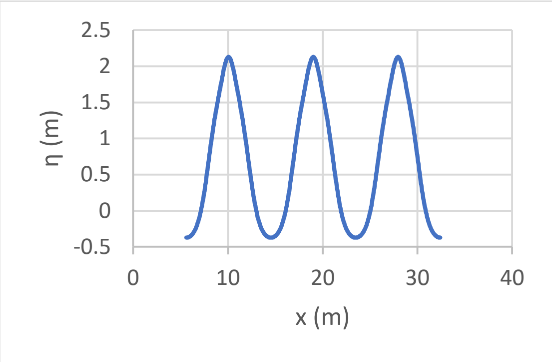

Fig.2. Wave profile with wave period of 8 sec., in3 wave lengths.

7.2. Comparison with Wiegel equation

Using data from an observation, Wiegel (1949-1964) formulated relation between wave period $T$and maximum wave height H_{max}in a wave period, i.e.

$$T_{Wieg} = 15.6\sqrt{\frac{H_{max}}{g}} \quad ....(58)$$

Table.5: Comparison with Wiegel equation

T	$\gamma = 3.0$, $\gamma_z = 1.63164$		$\gamma = 2.97102$, $\gamma_z = 1.60095$	
(sec)	H_{max} (m)	T_{Wieg} (sec)	H_{max} (m)	T_{Wieg} (sec)
6	1,40943	5,91305	1,45118	6
7	1,9184	6,89858	1,97523	7,00002
8	2,50569	7,88412	2,57992	8,00007
9	3,1713	8,86969	3,26522	9,00007
10	3,91518	9,85522	4,03119	10,0002
11	4,73746	10,8408	4,87774	11,0002
12	5,63796	11,8264	5,80504	12,0003
13	6,61695	12,8121	6,81286	13,0004
14	7,67409	13,7976	7,90155	14,0006
15	8,80954	14,7831	9,07066	15,0006

The comparison was done by calculating T_{Wieg}in (58) usingthe wave height which is the result of a calculation using the model, where the input in the model is wave period $T$and wave amplitude calculated using (52), so that the wave height that is obtained is the wave height maximumH_{max}in the related wave period.

Table (5) shows that for$\gamma = 3$, the obtained $T_{Wieg}$is almost similar with the Tthat is a wave period to calculate $H_{max}$with the model. Whereas in $\gamma = 2.97102$, it can be said that the obtained $T_{Wieg}$is equal with T. The result of this calculation concludes that the values of γ, $\gamma_z$and equations formulated in this research are in line with the result of Wiegel research (1949-1964) which is the result of an observation.

VIII. CONCLUSION

If the characteristic of ideal fluid i.e. irrotational flow is done at Euler momentum equation, and the velocity potential as the product of Laplace equation solution is substituted, the hydrodynamic force or convective equation in the horizontal direction becomes zero. This problem can be solved using weighted total acceleration where there is weighting coefficient at the differential term against vertical-z axis and the resulted model producewave height that corresponds to Wiegel equation.

Another finding that should be noticed is that the value of wave height is not twice the value of wave amplitude.

A further research needed is formulating shoaling and breaking model by doing the weighted total acceleration equation, because there are many researches result in the laboratory on breaker height that are stated in the form of Breaker Height Index equation, so that the shoaling-breaking model and its various basic theories can easily be calibrated.

REFERENCES

[1] Dean, R.G., Dalrymple, R.A. (1991). Water wave mechanics for engineers and scientists. Advance Series on Ocean Engineering.2. Singapore: World Scientific. ISBN 978-981-02-0420-4. OCLC 22907242.

[2] Hutahaean , S. (2019a). Application of Weighted Total Acceleration Equation on Wavelength Calculation. International Journal of Advance Engineering Research and Science (IJAERS). Vol-6, Issue-2, Feb-2019. ISSN-2349-6495(P)/2456-1908(O). https://dx.doi.org/10.22161/ijaers.6.2.31

[3] Hutahaean , S. (2019b). Water Wave Modeling Using Complete Solution of Laplace Equation. International Journal of Advance Engineering Research and Science (IJAERS). Vol-6, Issue-8, Aug-2019. ISSN-2349-6495(P)/2456-1908(O). https://dx.doi.org/10.22161/ijaers.6.8.33

[4] Courant, R.,Friedrichs,K., Lewy, H. (1928). Uber die partiellen Differenzengleic hungen der mathemtischen Physik. Matematische Annalen (in German). 100 (1): 32-74,

https://dx.doi.org/10.22161/ijaers.610.21 *ISSN: 2349-6495(P) | 2456-1908(O)*

Bibcode:1928. MatAn 100.32.C. doi :10.1007/BF01448839.JFM 54.0486.01 MR 1512478.

[5] Michell, J.H. (1893). On the highest wave in water: Philosofical Magazine, (5), vol. XXXVI, pp. 430-437.

[6] Wilson, B.W., (1963). Condition of Existence for Types of Tsunami waves, paper presented at XIII th General

[7]Wiegel,R.L. (1949). An Analysisis of Data from Wave Recorders on the Pacific Coast of tht United States, Trans.Am. Geophys. Union, Vol.30, pp.700-704.

[8]Wiegel,R.L. (1964). Oceanographical Engineering, Prentice-Hall, Englewoods Cliffs, N.J.