Boussinesq-type Equation Formulated using the Weighted Taylor Series

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Abstract— In this study, a Boussinesq equation was developed to model water waves. The equations were formulated using a weighted Taylor series, the Taylor series with weighted coefficients in the terms. The equation consists of the water surface elevation equation, the water particle velocity in the direction of the horizontal axis and vertical axis. The formulation of the continuity equation and the momentum equation is carried out on the same control volume with the same characteristics of particle velocity change. Thus, both equations have velocity variables with the same characteristics. The study obtained the Boussinesq equation that can be used with large wave amplitudes that are in line with those found in nature and can model shoaling and breaking well.

I. INTRODUCTION

The time series water wave model is generally referred to as the Boussinesq equation. There are many versions of the Boussinesq equation, both the continuity equation or the water surface equation and the momentum equation. Among researchers who developed the Boussinesq equationareBoussinesq (1871), Dingermans(1997), Hamm, Madsen, and Peregrin(1993), Johnson (1997), Kirby (2003), Peregrine (1967), Peregrine (1972), and many more. These equations are formulated using continuity equation and Euler'sthe momentum equation, which are well known, but in different explanationsmaking each researcher has his own Boussinesq equation.

The Boussinesq equations in this study were formulated using a different method from the previous Boussinesq equation formulation. This study formulated the continuity equation and momentum equation using the weighted Taylor series (Hutahaean, 2021a). The total acceleration in the momentum equation is formulated using the same control volume as in the continuity equation, where the horizontal velocity of x-direction only changes on the x-direction and the vertical velocity of z-direction changes only on the z-axis of vertical.

The basic form of the continuity equation obtained is the sum of the total acceleration in the horizontal-$x$direction with the total acceleration in the vertical z-axis. Furthermore, by working on very small time intervals, the term acceleration versus time (localacceleration) is omitted, obtaining a continuity equation in the form of a spatial differential only, as is the case with the continuity equation that has been widely used. However, there is a weighting coefficient on the horizontal differential term.

Furthermore, by reducing the spatial size of the control volume in the horizontal x-direction and in the vertical-$z$direction, the relation between the local acceleration of the particle in the vertical z-axis and the local acceleration of the particle in the horizontal x-direction is obtained. This relation is used to calculate the particle velocity in the vertical z-direction. While the particle velocity equation in the horizontal x-directionis obtained from the momentum equation in the horizontal x-direction. The hydrodynamic pressure and dynamic driving force in the horizontal axis-$x$are obtained by integrating the vertical z-direction momentum equation on the vertical axis.

By using the same control volume in the formulation of the continuity equation with the formulation of the momentum

equation, the definitive equation equal to the velocity of the particlewas obtained. Hence, there is a harmony between the continuity equation and the momentum equation.

Continuity Equation Formulation

The continuity equation is formulated by working on the principle of conservation of mass in an incompressible flow using the control volume in Figure (1). The input-output due to fluid flow in the control volume is as follows.

Input,

$I = \rho u\, \delta z + \rho w\, \delta x$

Output,

$O = \rho(u + \delta u)\delta z + \rho(w + \delta w)\delta x$

As a result of the input and output, at the small time interval δt, there is a change in the fluid mass at the control volume of,
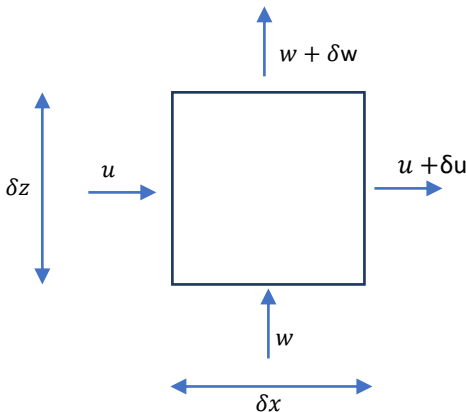
$\delta m = (I - O)\delta t$

Fig.1: Input-output on the controlvolume

Substitute the input and output equations, and divide both sides of the equation by$\delta t\, \delta x\, \delta z$,

$$\frac{\delta m}{\delta t\delta x\, \delta z} = -\rho\, \frac{\delta u}{\delta x} - \rho\, \frac{\delta w}{\delta z}$$

For a constant control volume, the mass change in the control volume is

$\delta m = \delta\rho\, \delta x\, \delta z$

equation obtained:

$$\frac{\delta\rho}{\delta t} = -\rho\, \frac{\delta u}{\delta x} - \rho\, \frac{\delta w}{\delta z}$$

for incompressible flow$\frac{\delta\rho}{\delta t} = 0$,

$$\frac{\delta u}{\delta x} + \frac{\delta w}{\delta z} = 0 \quad \dots\dots(1)$$

Equation (1) is a continuity equation in a very basic form. The derivative form of (1) is determined by the definitions of δu and δw. To get the form of δu and δw, the weighted Taylor series (Hutahaean, 2021a) is used for a function:

$$f(x, z, t) = \cosh k(h + z)\cos kx \cos\sigma t$$

k is the wave number, h is the water depth, σ is the angular frequency, x is the horizontal axis, z is the vertical axis, and t is the time. This equation is a form of velocity potential solution to Laplace's equation (Dean, 1991).$\cos\sigma t$ is used instead of $\sin\sigma t$. The weighted Taylor series from the equation is:

$$f(x + \delta x, z + \delta z, t + \delta t) = f(x, z, t)$$
$$+\gamma^2\, \delta t\, \frac{\mathrm{d}f}{\mathrm{d}t} + \gamma\, \delta x\, \frac{\mathrm{d}f}{\mathrm{d}x} + \delta z\, \frac{\mathrm{d}f}{\mathrm{d}z}$$

..(2)

$\gamma$is the weighting coefficient. In (2), there is a relation,

$$\delta z = \gamma\, \delta x \qquad \dots\dots(3)$$

For the horizontal velocity$u(x, z, t)$, the only thing that change is the horizontal x-direction,

$$u(x + \delta x, z, t + \delta t) = u(x, z, t)$$
$$+\gamma^2\, \delta t\, \frac{\mathrm{d}u}{\mathrm{d}t} + \gamma\, \delta x\, \frac{\mathrm{d}u}{\mathrm{d}x}$$

The first term of right-hand side is moved to the left and defined,

$$\delta u = \gamma^2\delta t\, \frac{\mathrm{d}u}{\mathrm{d}t} + \gamma\, \delta x\frac{\mathrm{d}u}{\mathrm{d}x}\dots..(4)$$

For the vertical velocity $w(x, z, t)$, which changes in the vertical z-direction only,

$$w(x, z + \delta z, t + \delta t) = w(x, z, t)$$
$$+\gamma^2\, \delta t\, \frac{\mathrm{d}w}{\mathrm{d}t} + \delta z\, \frac{\mathrm{d}w}{\mathrm{d}z}$$

δw is

$$\delta w = \gamma^2\, \delta t\, \frac{\mathrm{d}w}{\mathrm{d}t} + \delta z\, \frac{\mathrm{d}w}{\mathrm{d}z}\dots.(5)$$

Substitute (4) and (5) to (1),

$$\frac{\gamma^2\, \delta t\, \frac{\mathrm{d}u}{\mathrm{d}t} + \gamma\, \delta x\, \frac{\mathrm{d}u}{\mathrm{d}x}}{\delta x} + \frac{\gamma^2\, \delta t\, \frac{\mathrm{d}w}{\mathrm{d}t} + \delta z\, \frac{\mathrm{d}w}{\mathrm{d}z}}{\delta z} = 0$$

Substitute$\delta z$by (3)in the denominator of the second term and multiplying the equation by$\gamma\, \delta x$,

$$\gamma^3\delta t\frac{\mathrm{d}u}{\mathrm{d}t} + \gamma^2\, \delta x\, \frac{\mathrm{d}u}{\mathrm{d}x} + \gamma^2\, \delta t\, \frac{\mathrm{d}w}{\mathrm{d}t} + \delta z\, \frac{\mathrm{d}w}{\mathrm{d}z} = 0$$

…(6)

Then, with constantδxandδz, $\delta t$is reduced to close to zero, the equation becomes,

$$\gamma^2 \, \delta x \, \frac{\mathrm{d}u}{\mathrm{d}x} + \, \delta z \, \frac{\mathrm{d}w}{\mathrm{d}z} = 0$$

Substitute δz by (3)and divide the equation by $\gamma \, \delta x$,

$$\gamma \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}w}{\mathrm{d}z} = 0 \dots.(7)$$

(7) is theweighted continuity equationthat will be used to formulate the water surface elevation equation, in the following section.

II.   THE FORMULATION OF WATER SURFACE EQUATION $\eta(x,t)$.

The continuity equation (7) is multiplied by $dz$and integrated over water depth,

$$\gamma \int_{-h}^{\eta} \frac{\mathrm{d}u}{\mathrm{d}x} dz + \int_{-h}^{\eta} dw = 0$$

The integration of the 1st term is accomplished by the Leibniz integration rule. The equation of the Leibniz integral rule (Protter, Murray, Morrey, &Charles, 1985)is:

$$\int_{\alpha}^{\beta} \frac{\mathrm{d}f}{\mathrm{d}x} dz = \frac{\mathrm{d}}{\mathrm{d}x}\int_{\alpha}^{\beta} u \, dz - f_{\beta}\frac{\mathrm{d}\beta}{\mathrm{d}x} + f_{\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}x} \dots\dots(8)$$

Then the integration of the 1st term is,

$$\gamma \int_{-h}^{\eta} \frac{\mathrm{d}u}{\mathrm{d}x} dz = \gamma \frac{\mathrm{d}}{\mathrm{d}x}\int_{-h}^{\eta} u \, dz - \gamma u_{\eta}\frac{\mathrm{d}\eta}{\mathrm{d}x} - \gamma u_{-h}\frac{\mathrm{d}h}{\mathrm{d}x}$$

where $\frac{\mathrm{d}(-h)}{\mathrm{d}x} = -\frac{\mathrm{d}h}{\mathrm{d}x}$, $u_{\eta}$is the horizontal velocity at the surface, while $u_{-h}$is the horizontal velocity on the sea bed. This integration is substituted into(8), while w_{η}particle velocity of z-direction is substituted by weighted Kinematic Free Surface Boundary Condition (Hutahaean, 2021a), which is:

$$w_{\eta} = \gamma \frac{\mathrm{d}\eta}{\mathrm{d}t} + u_{\eta}\frac{\mathrm{d}\eta}{\mathrm{d}x},$$

So,

$$\gamma \frac{\mathrm{d}}{\mathrm{d}x}\int_{-h}^{\eta} u dz - \gamma u_{\eta}\frac{\mathrm{d}\eta}{\mathrm{d}x} - \gamma u_{-h}\frac{\mathrm{d}h}{\mathrm{d}x}$$
$$+\gamma \frac{\mathrm{d}\eta}{\mathrm{d}t} + u_{\eta}\frac{\mathrm{d}\eta}{\mathrm{d}x} - w_{-h} = 0$$

Integration is completed by working on the concept of depth average velocity, while $w_{-h}$is substituted with the bottom kinematic boundary condition,

$$\gamma \frac{\mathrm{d}\eta}{\mathrm{d}t} + \gamma \frac{\mathrm{d}\beta_u UH}{\mathrm{d}x} + (1-\gamma)\alpha_{u\eta}U\frac{\mathrm{d}\eta}{\mathrm{d}x} + (1-\gamma)\alpha_{uh}U\frac{\mathrm{d}h}{\mathrm{d}x}$$
$$= 0$$

…(9)

$U$is the horizontal depth average velocity, $\beta_u$is the coefficient integration, $H$is the total water depth $H = h + \eta$, while $\alpha_{u\eta}$is the transformation coefficient from the horizontal surface velocity $u_{\eta}$to the depth average velocity U, while $\alpha_{uh}$is the transformation coefficient from $u_{-h}$to

depth average velocity U.These coefficients will be discussed in section 4.

III.   EQUATION OF PARTICLE VELOCITY IN THE HORIZONTAL DIRECTION AND VERTICAL DIRECTION

3.1. Momentum equation

For the conformity with the continuity equation, the momentum equation will be formulated using the control volume in figure (1) where the horizontal velocity u only changes on the horizontal x-axis, while the vertical velocity w only changes on the vertical z-axis. Equation (4) is divided by δt and very small δt and δxclose to zero were taken,

$$\frac{Du}{dt} = \gamma^2 \frac{\mathrm{d}u}{\mathrm{d}t} + \gamma \, u \frac{\mathrm{d}u}{\mathrm{d}x} \qquad \dots\dots(10)$$

This equation is the total acceleration in the horizontal x-direction. Equation (5) is divided by δt and very small $\delta t$and δzclose to zero were taken,

$$\frac{Dw}{Dt} = \gamma^2 \frac{\mathrm{d}w}{\mathrm{d}t} + w \frac{\mathrm{d}w}{\mathrm{d}z} \qquad \dots\dots(11)$$

This equation is the total acceleration in the vertical z-direction.

A method is used to formulate the Euler's momentum equationin (10) and (11),that the mass multiplied by the acceleration is the same as the driving force, the equations for the horizontal x-directionand vertical z-direction are obtained, respectively,

$$\gamma^2 \frac{\mathrm{d}u}{\mathrm{d}t} + \gamma \, u \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} \qquad \dots\dots(12)$$
$$\gamma^2 \frac{\mathrm{d}w}{\mathrm{d}t} + w \frac{\mathrm{d}w}{\mathrm{d}z} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}z} - g \dots\dots(13)$$

3.2. Hydrodynamic Pressure Equation

To obtain the pressure equation p, (13) is multiplied by $dz$and integrated about the vertical z-axis,

$$\gamma^2 \int_{z}^{\eta} \frac{\mathrm{d}w}{\mathrm{d}t} \, dz + \frac{1}{2}\int_{z}^{\eta} dww = -\frac{1}{\rho}\int_{z}^{\eta} dp - g \int_{z}^{\eta} dz$$

The integration of the 2ndterm on the left and the 1st and 2nd terms on the right is solved.The surface dynamic boundary conditions are done $p_{\eta} = 0$and the equation is written as the equation for p,

$$\frac{p}{\rho} = \gamma^2 \int_{z}^{\eta} \frac{\mathrm{d}w}{\mathrm{d}t} \, dz + \frac{1}{2}\left(w_{\eta}w_{\eta} - ww\right) + g(\eta - z)$$
…..(14)

This equation is the hydrodynamic pressure equation. Next, (14) is differentiated about the horizontal x-axis,

$$\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} = \gamma^2 \frac{\mathrm{d}}{\mathrm{d}x}\int_{z}^{\eta} \frac{\mathrm{d}w}{\mathrm{d}t} \, dz + \frac{1}{2}\frac{\mathrm{d}w_{\eta}w_{\eta}}{\mathrm{d}x} - \frac{1}{2}\frac{\mathrm{d}ww}{\mathrm{d}x} + g \frac{\mathrm{d}\eta}{\mathrm{d}x}$$

Hutahaean (2021b) solves the integration and differential on the 1st term on the right side of the equation by using the velocity potential of the solution to the Laplace equation obtaining

$$\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} = \gamma^2\left(\frac{\mathrm{d}u_\eta}{\mathrm{d}t} - \frac{\mathrm{d}u}{\mathrm{d}t}\right) + \frac{1}{2}\frac{\mathrm{d}w_\eta w_\eta}{\mathrm{d}x} - \frac{1}{2}\frac{\mathrm{d}ww}{\mathrm{d}x} + g\frac{\mathrm{d}\eta}{\mathrm{d}x}$$

……..(15)

This equation is the equation for the hydrodynamic driving force in the horizontal x-direction.

3.3. Velocity Equation in the Horizontal x-direction

(15) is substituted to (12),

$$\gamma^2\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\gamma}{2}\frac{\mathrm{d}uu}{\mathrm{d}x} = -\gamma^2\left(\frac{\mathrm{d}u_\eta}{\mathrm{d}t} - \frac{\mathrm{d}u}{\mathrm{d}t}\right)$$
$$- \frac{1}{2}\frac{\mathrm{d}w_\eta w_\eta}{\mathrm{d}x} + \frac{1}{2}\frac{\mathrm{d}ww}{\mathrm{d}x} - g\frac{\mathrm{d}\eta}{\mathrm{d}x}$$

The 1st term on the left side and the the 2nd term on the right side cancel each other out, then this equation is multiplied by dz and integrated over water depth,

$$\gamma^2\frac{\mathrm{d}u_\eta}{\mathrm{d}t}H + \frac{\gamma}{2}\int_{-h}^{\eta}\frac{\mathrm{d}uu}{\mathrm{d}x}dz = -\frac{1}{2}\frac{\mathrm{d}w_\eta w_\eta}{\mathrm{d}x}H$$
$$+ \frac{1}{2}\int_{-h}^{\eta}\frac{\mathrm{d}ww}{\mathrm{d}x}dz - g\frac{\mathrm{d}\eta}{\mathrm{d}x}H \;\dots\dots(16)$$

The integration of the 2nd term on the left and right sides is solved by the Leibniz Integral rule,

$$\int_{-h}^{\eta}\frac{\mathrm{d}uu}{\mathrm{d}x}dz = \frac{\mathrm{d}}{\mathrm{d}x}\int_{-h}^{\eta}uu\,dz - u_\eta u_\eta\frac{\mathrm{d}\eta}{\mathrm{d}x} - u_{-h}u_{-h}\frac{\mathrm{d}h}{\mathrm{d}x}$$

The integration of the right-hand side is solved by using the concept of depth average velocity,

$$\int_{-h}^{\eta}\frac{\mathrm{d}uu}{\mathrm{d}x}dz = \frac{\mathrm{d}\beta_{uu}UUH}{\mathrm{d}x} - u_\eta u_\eta\frac{\mathrm{d}\eta}{\mathrm{d}x} - u_{-h}u_{-h}\frac{\mathrm{d}h}{\mathrm{d}x}$$

β_{uu} is the coefficient integration, that will be discussed in section 4, then the horizontal velocity at the surface u_η and the horizontal velocity on the sea bed u_{-h} are transformed into horizontal depth average velocity,

$$\int_{-h}^{\eta}\frac{\mathrm{d}uu}{\mathrm{d}x}dz = \frac{\mathrm{d}\beta_{uu}UUH}{\mathrm{d}x} - \alpha_{u\eta}\alpha_{u\eta}UU\frac{\mathrm{d}\eta}{\mathrm{d}x}$$
$$- \alpha_{uh}\alpha_{uh}UU\frac{\mathrm{d}h}{\mathrm{d}x}$$

$\alpha_{u\eta}$ and α_{uh} are transformation coefficients that will be discussed in section 4.

In the same way, it will be obtained,

$$\int_{-h}^{\eta}\frac{\mathrm{d}ww}{\mathrm{d}x}dz = \frac{\mathrm{d}\beta_{ww}WWH}{\mathrm{d}x} - \alpha_{w\eta}\alpha_{w\eta}WW\frac{\mathrm{d}\eta}{\mathrm{d}x}$$
$$- w_{-h}w_{-h}\frac{\mathrm{d}h}{\mathrm{d}x}$$

β_{ww} coefficient integration and $\alpha_{w\eta}$ is the transformation coefficient. Kinematic sea bed boundary condition is $w_{-h} = -u_{-h}\frac{\mathrm{d}h}{\mathrm{d}x}$. So

$$w_{-h}w_{-h}\frac{\mathrm{d}h}{\mathrm{d}x} = -u_{-h}u_{-h}\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^3$$

$\frac{\mathrm{d}h}{\mathrm{d}x}$ is the small bottom slope, so $w_{-h}w_{-h}\frac{\mathrm{d}h}{\mathrm{d}x}$ is very small and can be ignored, so

u_η in (16) is transformed into U and w_η is transformed into W and divided by H,

$$\gamma^2\alpha_{u\eta}\frac{\mathrm{d}U}{\mathrm{d}t} + \frac{\gamma}{2H}\int_{-h}^{\eta}\frac{\mathrm{d}uu}{\mathrm{d}x}dz = -\frac{\alpha_{w\eta}\alpha_{w\eta}}{2}\frac{\mathrm{d}WW}{\mathrm{d}x}$$
$$+ \frac{1}{2H}\int_{-h}^{\eta}\frac{\mathrm{d}ww}{\mathrm{d}x}dz - g\frac{\mathrm{d}\eta}{\mathrm{d}x}$$

The result of integration is substituted $\int_{-h}^{\eta}\frac{\mathrm{d}uu}{\mathrm{d}x}dz$ and $\int_{-h}^{\eta}\frac{\mathrm{d}ww}{\mathrm{d}x}dz$

$$\gamma^2\alpha_{u\eta}\frac{\mathrm{d}U}{\mathrm{d}t} + \frac{\gamma}{2H}\frac{\mathrm{d}\beta_{uu}UUH}{\mathrm{d}x}$$
$$- \frac{\gamma\alpha_{u\eta}\alpha_{u\eta}}{2H}UU\frac{\mathrm{d}\eta}{\mathrm{d}x} - \frac{\gamma\alpha_{uh}\alpha_{uh}}{2H}UU\frac{\mathrm{d}h}{\mathrm{d}x} =$$
$$- \frac{\alpha_{w\eta}\alpha_{w\eta}}{2}\frac{\mathrm{d}WW}{\mathrm{d}x} + \frac{1}{2H}\frac{\mathrm{d}\beta_{ww}WWH}{\mathrm{d}x}$$
$$- \frac{\alpha_{w\eta}\alpha_{w\eta}}{2H}WW\frac{\mathrm{d}\eta}{\mathrm{d}x} - g\frac{\mathrm{d}\eta}{\mathrm{d}x}\dots..(17)$$

(17) is the equation for the particle velocity in the horizontal x-direction.

3.4. Velocity Equation vertical z-direction

In (6), with constant δt, the size of δx and δz are reduced to a point, so that the 2nd and 4th terms become zero, then divided by $\gamma^2\,\delta t$, then the equation becomes,

$$\gamma\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\mathrm{d}w}{\mathrm{d}t} = 0$$

This equation is written as the equation for $\frac{\mathrm{d}w}{\mathrm{d}t}$ multiplied by dz and integrated about the vertical z-direction. The integration is completed by Leibniz integral rule and the concept of depth average velocity is obtaining

$$\beta_w H\frac{\mathrm{d}W}{\mathrm{d}t} + (\beta_w - \alpha_{w\eta})W\frac{\mathrm{d}\eta}{\mathrm{d}t} =$$
$$-\gamma\beta_u H\frac{\mathrm{d}U}{\mathrm{d}t} + \gamma(\beta_u - \alpha_{u\eta})U\frac{\mathrm{d}\eta}{\mathrm{d}t} \qquad \dots\dots(18)$$

In this equation $\frac{\mathrm{d}U}{\mathrm{d}t}$ is obtained from (17) while $\frac{\mathrm{d}\eta}{\mathrm{d}t}$ is obtained from (9). The equation(18) can be said to be an expression of the law of conservation of energy, where there is an interaction of $\frac{\mathrm{d}U}{\mathrm{d}t}$, $\frac{\mathrm{d}W}{\mathrm{d}t}$ and $\frac{\mathrm{d}\eta}{\mathrm{d}t}$.

IV.   COEFFICIENT INTEGRATION AND TRANSFORMATION COEFFICIENT

In the previous sections, the integration of water depth is done using the concept of depth average velocity, where the depth average velocity (Dean, 1991) is,

$$U(x,t) = \frac{1}{\beta_u H}\int_{-h}^{\eta} u\, dz \quad \ldots\ldots(19)$$

β_u =Coefficient of integration of horizontal velocity u, and$H = h + \eta$

Furthermore, in this study, it is defined that the depth average velocity is the velocity at a depth of $z = z_0$ (Fig. 2), and the integration coefficient β_u is,
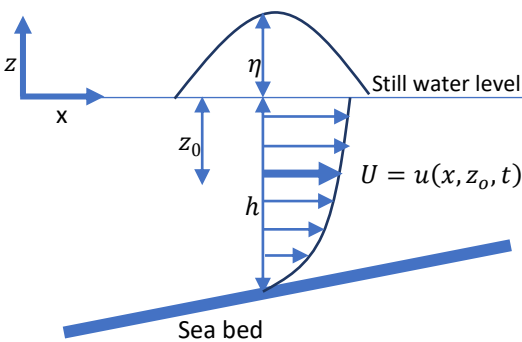
$$\beta_u = \frac{1}{Hu(x,z_0,t)}\int_{-h}^{\eta} u\, dz$$

Fig 2. Depth average velocity

Using the particle velocity equation from the velocity potential,

$$u(x,z,t) = Gk\cosh k(h+z)\cos kx \sin \sigma t$$

So,

$$\frac{u}{U} = \frac{\cosh k(h+z)}{\cosh k(h+z_0)}$$

$$\beta_u = \frac{1}{H\cosh k(h+z_0)}\int_{-h}^{\eta}\cosh k(h+z)\, dz$$

Integration is completed and by taking $\eta = \frac{A}{2}$, so

$$\beta_u = \frac{\sinh k\left(h+\frac{A}{2}\right)}{kH\cosh k(h+z_0)}\ldots..(20)$$

Law of conservation of wave number (Hutahaean. 2021),

$$\frac{\mathrm{d}k\left(h+\frac{A}{2}\right)}{\mathrm{d}x} = 0\ldots..(21)$$

where$k\left(h+\frac{A}{2}\right) = kH = constant = \theta\pi$, so

$$\beta_u = \frac{\sinh \theta\pi}{\theta\pi \cosh k(h+z_0)}\ldots..(22)$$

So to get β_u is to set the value ofz_0 beforehand.

θ is referred to as the deep water coefficient which is greater than or equal to 1 where $\tanh \theta\pi \approx 1$. This coefficient for defining the deep water limit, for example SPM (1984), using$\theta = 1$, in this case, deep water depth is$h_0 = \frac{L_0}{2}$. In this study,$\theta = 1.8$ is used, this value was obtained based on a study of the solution generated by the model.

$$\int_{-h}^{\eta} w\, dz = \beta_w WH$$

$$\beta_w = \frac{1}{H\sinh k(h+z_0)}\int_{-h}^{\eta}\sinh k(h+z)\, dz$$

By completing the integration obtaining,

$$\beta_w = \frac{\cosh \theta\pi - 1}{\theta\pi \sinh k(h+z_0)}\ldots.(23)$$

The other integration coefficients are,

$$\int_{-h}^{\eta} UU\, dz = \beta_{uu}UUH$$

$$\beta_{uu} = \frac{1}{H\cosh^2 k(h+z_0)}\int_{-h}^{\eta}\cosh^2 k(h+z)\, dz$$

By completing the integration obtaining,

$$\beta_{uu} = \frac{\left(\frac{1}{2}\sinh 2\theta\pi + \theta\pi\right)}{2\,\theta\pi\cosh^2 k(h+z_0)}\ldots.(24)$$

$$\int_{-h}^{\eta} WW\, dz = \beta_{ww}WWH$$

$$\beta_{ww} = \frac{1}{H\sinh^2 k(h+z_0)}\int_{-h}^{\eta}\sinh^2 k(h+z)\, dz$$

By completing the integration obtaining,

$$\beta_{ww} = \frac{\frac{1}{2}\sinh 2\theta\pi - \theta\pi}{2\theta\pi \sinh^2 k(h+z_0)}\ldots.(25)$$

The integration coefficients are calculated in deep water. The value of the integration coefficient is constant, considering (21).

In the momentum equation and the continuity equation, there are surface velocities u_η andw_η and seabed velocities u_{-h} andw_{-h}.   These variables must be transformed into depth average velocity of U and$W$, with the following transformation coefficients.

$$u_\eta = \frac{\cosh \theta\pi}{\cosh k(h+z_0)}U$$

$$u_\eta = \alpha_{u\eta} U$$

$$\alpha_{u\eta} = \frac{\cosh\theta\pi}{\cosh k(h+z_0)}....(26)$$

$$u_{-h} = \alpha_{uh}\, U$$

$$\alpha_{uh} = \frac{1}{\cosh k(h+z_0)}...(27)$$

$$w_\eta = \alpha_w\, W$$

$$\alpha_{w\eta} = \frac{\sinh\theta\pi}{\sinh k(h+z_0)}...(28)$$

The vertical velocity transformation on the sea bed cannot be used for potential flow equations because it will produce a vertical bottom velocity which is equal to zero. For this reason, the vertical bottomvelocity is changed to horizontal bottom velocity by using the kinematic bottom boundary condition, which is,

$$w_{-h} = -u_{-h}\frac{dh}{dx}$$

$\frac{dh}{dx}$ is bottom slope, so

$$w_{-h} = -\alpha_{uh} U\frac{dh}{dx}...(29)$$

V. MODEL RESULTS

First, the model is carried out on a channel with a constant depth, with a water depth of $h = 15.0\, m$ and a canal length of 300 m. The input is a sinusoidal wave with period $T = 8$ second and wave amplitude $A = 1.20\, m$. The calculation constants used are $\gamma = 1.6$, $\theta = 1.8$, and $z_0 = -0.4\, h$. The model is executed up to 11 times the wave period. The model outputs are presented in Figure 3 and Figure 4.

In the output of the model, the resulting solution is stable at the execution of 11 times the wave period. This shows the stability of the equation and the numerical method. The wavelength of the model is quite short, around 25 m (Figure 3 and Figure 4). The particle velocity in the vertical direction W is greater than the particle velocity in the horizontal direction U with the opposite phase. The horizontal phase velocity U is the same as the water surface movement phase η while the particle phase velocity in the vertical W-direction is opposite to both. This shows that the change in kinetic energy in the vertical direction is proportional to the amount of change in horizontal kinetic energy with changes in potential energy at the water surfaceelevation.

In addition, when $U = 0$ and $\eta = 0$, W is also equal to zero. This makes it easier to define the initial condition where at $t = 0$, $U = \eta = W = 0$. Calculation of the water particlevelocity in the vertical direction using the Kinematic Free surface Boundary Condition, there will be difficulties in defining the initial value, considering that this equation produces a phase where at $U = 0$ and $\eta = 0$,

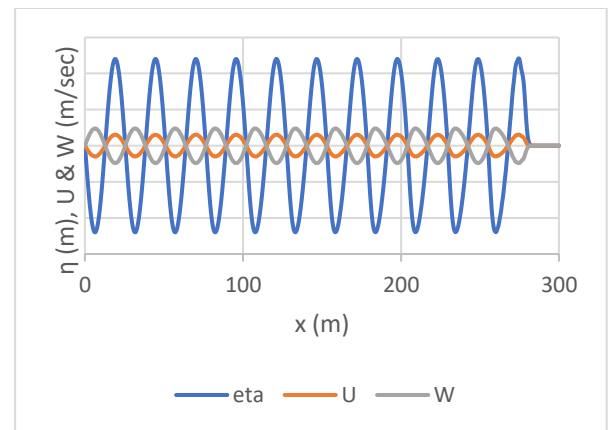the particle velocity in the vertical W-direction is in the maximum phase.

Fig 3. Output model on theflat bottom.
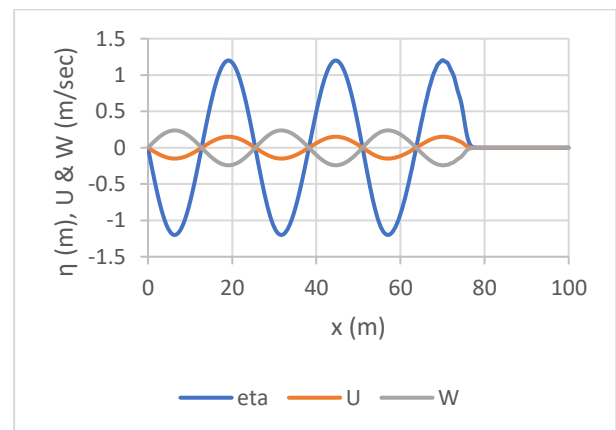

Fig 4. Output model on theflat bottom.

Next, the model is executed on a channel with a bottom slope of -0.07. The upstream water depth is 15.0 m, while the downstream water depth is 1.0 m, with a channel length of 200 m. The incoming wave is a sinusoidal wave with a wave period of $T = 8\, sec$, with an initial wave amplitude of $A_0 = 1.2\, m$. The model output is presented in Fig 5.
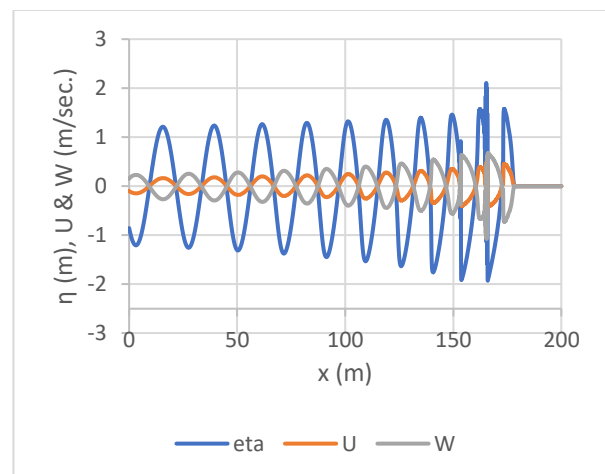

Fig 5. Output model on the sloping bottom.

Fig 5 shows that the model can model the phenomenon of shoaling well. At a distance of $x = 150.0\ m$, at a water depth of $h = 4.50\ m$, the wave begins to break, followed by the main breaking at $h = 3.80\ m$ with a breaking amplitude of $A_b = 1.5$. Furthermore, after breaking, the model stops after one wave period. From these results, it can be said that the model can simulate shoaling with breaking even though it is not complete until the waves disappear.

VI. CONCLUSION

The formulation of the total acceleration equation in the momentum equation using the same control volume as in the continuity equation provides certainty that the velocity characteristics in the momentum equation and the continuity equation are the same. Thus, it can be said that the velocity stated in the continuity equation is exactly the generated velocity by the momentum equation creating a good couple between the continuity equation and the momentum equation.

By defining that the depth average velocity is a representative velocity of the velocity at a certain depth, the integration coefficient, and the transformation coefficient can be calculated using the velocity potential theory.

The weighted coefficient on the Taylor series functions to adjust the particlewavelength and velocity, be it horizontal or vertical. The greater the value of the weighted coefficient, the shorter the wavelength, the smaller the particle velocity in the horizontal direction while the particle velocity in the vertical direction will be greater.

Determination of the deepwater depth represented by the deepwater coefficient, in addition to determining the stability of the solution, also determines the depth at which breaking begins to occur.

In general, the model gives good results where the model can be carried out at large wave amplitude that match those in nature, and the model can simulate shoaling and breaking well. However, it requires development so that the model can simulate breaking completely with the wave height that gradually gets smaller until it disappears after breaking.

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