Deformation Contour Analysis in Elastic Plastic Fatigue Fissure using Local and Global Integral Approach

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Abstract— This paper present mathematical modelof local and global integral for elastic plastic damage mechanism of displacement discontinuity. The discrete nature of stress intensity factor approach where there is large scale yielding is inadmissible; hence, continuous energy path contour analysis becomes crucial. The simple energetic path can easily be evaluated computationally and the most prevalent method. However, fails to account for dynamic effect, kinetic energy and thermal effect, body force and effect of surface traction. The research is focused on deformation contour analysis of elastic-plastic void growth. In this paper, local andglobal representation of deformation field near a crack tip due to loading was examined using Divergence theorem and energy balanced method. The local integral is equivalent to energy release rate for nonlinear elastic materialunder quasi-static condition. The global J- Integral captured thermal effect, body force non zero surface traction and effect of plasticity. The integral permits more accurate analysis possible for all deformation fields including very close proximity of crack tip and all categories of material.TheFatigue crack growth analysis was developed to support economic fail safe and damage tolerance philosophies, and the execution, effectuation of damage tolerance control procedures for the workings of structural systems.

Keywords— Fissure, local integral, global integral, contour and deformation.

I. INTRODUCTION

Plane strain deformation field surrounding the tip of a crack is of immense significance in damage mechanism of solid component in structural design and analysis. It deals the studies of the behaviour of stress and strain field in the neighbourhood of fissure tip. The product of stress and strain exhibits a singularity which is inversely proportional to the magnitude of crack from the tip and near the tip where there is numerous invisible deformation fields or contours. This represents energy line integral with the characteristics of path that does not depend on all the contours encircling fissure tip. In Rice (1968) this line integral has same value for all paths around the tip of a crack in two dimensional deformation field of linear and non-linear elastic material. The correct choice of this contour is crucial; the result gives approximate estimate of the degree of strain concentration near the tip of a crack in a notch crack problem. Irwin (2004) posited that the principle of continuous deformation path is related to path independence if a path c_2 was obtained from c_1 by

continuously moving c_1 with ends fixed until it coincide with c_2. This revealed thatthere are two of the indefinite numerous intermediate paths for which the integral always retain its value. Besides, there is a continuous deformation path of an integral keeping the ends fixed since deformation path always consist of only points at which the function is analytic, hence the integral retain same value.

Contour integral has great contribution in fracture mechanics; it characterizes the parameter used for nonlinear materials by idealizing the elastic plastic deformation as nonlinear elastic.According to Boulenouar et al (2013) the use of crack propagation laws based on stress intensity factors is quite good in scientific and engineering application of mechanics of fracture. The stress intensity factor sufficiently defines the stress field close to the crack tip and provides fundamental information on how the crack is going to advance. Furthermore, Rice (1967) applied deformation plasticity to analyze crack in a nonlinear material; therefore, in nonlinear energy released rate, J integral could be used

and represented as a path independent line integral.Riceand Rosengren (1968) also showed that J integral uniquely characterize crack tip stress and strain in nonlinear materials. So,the J integral could be viewed as both an energy parameter and a stress intensity parameter. It is again seen as path independence contour integral for the analysis of crack problems. The contour integral could be evaluated along a contour surrounding the crack tip where numerical accuracy is better. In path independence, it is possible to compute J at remote contour provided an appropriate correction term that is an area integral is applied. Furthermore, for J integral to remain path independent, the stress strain must vary as $\dfrac{1}{r}$ near crack tip. This energy release rate measuresthe energy available for an increment of crack extension also known as crack driving force or crack extension force. It has the potential ability to evaluate the integrity of structural component: it has become common to evaluate the integrity of structures using path independent J- integral interpreted as the intensity of the elastic – plastic deformation and the stress field surrounding a crack tip. However, the integral loses the property of path independence and can no longer be interpreted as the energy released rate under some conditions. It is not evident that if the elastic plastic stress or strain fields near a crack tip is characterizable in terms of J-integral. In addition, Oyesanya (2007) found that the integral cannot characterize the crack tip stress strain analysis of structures under thermal loading outside HRR dominance. In small size dependent fracture analysis characterized by strain gradient elasticity the convenient strain gradient J- integral is incapable of accounting for the deformation energy distribution from the strain gradient. It underestimates the critical load and over estimates the critical deflection then higher order J- integral was necessary for analysis of fracture behaviour of structures with none negotiable strain gradient Lam e'tal (2004).

Due to complexities in investigating crack tip stress and strain of displacement amplitude loading condition, the discrete nature of stress intensity factor approach becomes inadmissible where there is large scale yielding.The need for continuous energy path contour becomes crucial. However, the simple energetic path fails to account for dynamic effect, kinetic energy, thermal effect and body force. It loses its physical significance as a crack driving force when deformation is not reversible. Therefore, in this paper, a global representation of deformation field near a crack tip

due to loading was examined using globalized energy balance approach. It is a general representation of J integral which includes the existence of a fracture process region and the effect of plastic deformation, body force, thermal strain and inertial effect of material.Crack tip singularities are investigated using energy line integral that exhibits path independence for all contours near the tip of the crack in a two dimensional deformation field of a linear elastic material. Besides, the global J- Integral is used to extract the magnitude of crack tip stress intensity factor in fatigue crack problem and the characteristics of the J-integral under mode-1 loading. The integral permits accurate analysis possible for all deformation fields including very close proximity of crack tip. This arbitrary continuous and differentiable contour represent the magnitude of energy available for crevice propagation, that is, the value of this integral equates the energy released rate of linear and nonlinear elastic plastic body which contains void.

II. LOCAL ENERGETIC PATH CONTOUR INTEGRAL

Plastic zone size plays crucial role in actual crack propagation, hence elastic fracture dynamics that can be used notably among is J-integral. It is a parameter that deals with energy while stress intensity factor is a local parameter that deals with displacement and stress field in the vicinity of a crack. The approaches may be different but the goal is the same to characterize a crack. The magnitude of plastic zone affects that actual fatigue crack progression. Therefore, elastic plastic fracture dynamic model for crack growth becomes crucial. This gives rise to energy flux integral that denotes path independent energy line integral that quantifies the strength of the singular stress and strain field in the vicinity of a crack tip (a measure of energy available for an increment of crack extension, the energy released rate). It is also known as crack extension force or crack driving force. The magnitude of this J-contour integral can be calculated from a path independent contour integral in the neighbourhood of the flaw tip.

However, taking into view an arbitrary contour *c*that encloses a crack tip whereby the path is in anticlockwise direction to evaluating J along the contour**,** the value of this integral called contour integral. This is equal to the energy released rate in a nonlinear elastic material that contains a crack.In Jin and Sun (2004), and Daniel (2003) potential energy of a crack body is denoted by

$$J = \frac{d\pi}{dA}$$

where π is the potential energy of body. The potential energy equals strain energy minus work done. The force induced by external agencies is given as

$$\pi = u - F$$

where u = the strain energy stored in the body and

F = work done by external forces.

$$G = J = -\frac{d\pi}{dA}$$

where G is the rate of change in potential energy with the crack,

A = crack area,

u = strain energy stored in the body and

F = work done by external forces.

$$\pi(l) = \iint_{A_0} w\,dx\,dy - \int_{c_t} T_i u_i\,dc \qquad (1)$$

π = potential energy of cracked solid,

$W =$ strain energy density

T_i = traction on the boundary segment

$$^{c_t}\,G = \frac{d\pi}{dl} = -\frac{d}{dl}\iint_{A_0} w\,dx\,dy + \frac{d}{dl}\int T_i u_i\,dc \qquad (2)$$

$$G = -\frac{d}{dl}\left(\iint_A w\,dx\,dy + \iint_{A_h} w\,dx\,dy\right) + \int_{c_t} T_i \frac{du_i}{dl}\,dc$$

$$G = -\iint_A \frac{dw}{dl}dx\,dy + \int T_i \frac{du_i}{dl}\,dc - \frac{d}{dl}\iint_{A_h} w\,dx\,dy \qquad (4)$$

$$G = -\iint_A \frac{dw}{dl}dx\,dy + \iint_A \frac{dw}{dx}\,dx\,dy + \int_{c_0} T_i \frac{\partial u_i}{\partial l}\,dc - \int T_i \frac{\partial u_i}{\partial x} - \frac{d}{dl}\iint_{A_0} w\,dx\,dy \qquad (5)$$

$$\int_{c_0} T_i \frac{\partial u_i}{\partial l}\,dc = \iint_{A_h} \frac{\partial w}{\partial l}dA$$

$$(6)$$

$$dA = rdrd\theta$$

$$A_h = A_0 - A$$

Substituting equation (6) into equation (5) hence integration along the contour c_0 is reduced to c

$T_i = \sigma_{ij}n_j = 0$ on the part of the contour c_0

$$G = \iint_A \frac{\partial w}{\partial x}dxdy - \int_c T_i \frac{\partial u_i}{\partial x}dc + \iint_{A_h} \frac{\partial w}{\partial l}dxdy - \frac{d}{dl}\iint_{A_h} wdxdy \qquad (7)$$

Using the Divergence Theorem and considering the first term of equation (7) gives below

$$\iint \frac{\partial w}{\partial x}dxdy = \int_c wdy + \int_{c_h} wdy \qquad (8)$$

Substitute equation (8) into equation (7)

$$G = \int_c wdy + \int_{c_h} wdy - \int_c T_i \frac{\partial u_i}{\partial x}dc + \iint_{A_h} \frac{\partial w}{\partial l}dxdy - \frac{d}{dl}\iint_{A_h} wdxdy \qquad (9)$$

$$w = B(l)\widetilde{w}(X - l, Y) = B(l)\widetilde{w}(x, y) \qquad (10)$$

$$\int_{A_h} wdy = \int_{-h}^{h} B(l)\widetilde{w}(-h, y)dy + \int_{h}^{-h} B(l)\widetilde{w}(h, y)dy \qquad (11)$$

$$\iint_{A_h} \frac{\partial w}{\partial l}dxdy = \iint_{A_h} B'(l)\widetilde{w}(x, y)dxdy \qquad (12)$$

Considering the last term in equation (5)

$$\frac{d}{dl}\iint_{A_h} wdxdy$$

$$x = X - l$$

$$w = B(l)\widetilde{w}(x - l, Y) = B(l)\widetilde{w}(x, y)$$

$$\frac{d}{dl}\iint_{A_h} wdxdy = \lim_{\Delta l \to 0} \frac{1}{\Delta l}\left[\iint_{A_h} (B(l + \Delta l)\widetilde{w}(x - \Delta l, y) - B(l)w(x, y))dxdy\right] \qquad (13)$$

Consider the dummy variable below,

$$x^* = x - \Delta l$$

$$\iint_{A_h} \widetilde{w}(x - \Delta l, y)dxdy = \int_{-h}^{h}\left(\int_{-h-\Delta l}^{h-\Delta l} w(x^*, y)dx^*\right)dy \qquad (14)$$

$$\iint_{A_h} \tilde{w}(x-\Delta l, y)dxdy = \int_{-h}^{h} \left(\int_{-h-\Delta l}^{h-\Delta l} w(x^*, y)dx^* + \int_{-h}^{h} w(x^*, y)dx^* + \int_{h}^{h-\Delta l} \tilde{w}(x^*, y)dx^* \right)dy \qquad (15)$$

$$\iint_{A_h} \tilde{w}(x-\Delta l, y)dxdy = \iint \tilde{w}(x, y)dxdy + \int_{-h}^{h} \left(\int_{-h-\Delta l}^{h-\Delta l} \tilde{w}(x, y)dx - \int_{h-\Delta l}^{h} \tilde{w}(x, y)dx \right)dy \qquad (16)$$

$$\Delta l \to 0$$

$$\iint_{A_h} \tilde{w}(x-\Delta l, y)dxdy = \iint_{A_h} \tilde{w}(x, y)dxdy + \int_{-h}^{h} (w(-h, y)\Delta l + \tilde{w}(h, y)\Delta l)dy \qquad (17)$$

As $\Delta l \to 0$ then,

$$B(l + \Delta l) = B(l) + B'(l)\Delta l \qquad (18)$$

Putting equation (13) and equation (18) into equation (13) gives

$$\frac{d}{dl} \iint_{A_h} W dxdy = B'(l) \iint_{A_h} \tilde{w}(x, y)dxdy + B(l) \int_{-h}^{h} (w'(-h, y) - w'(h, y))dy \qquad (19)$$

Substituting equation (11), equation (12) and equation (19) into equation (9)

$$G = \int_c w dy - \int_c T_i \frac{\partial u_i}{\partial x} ds = J \qquad (20)$$

This is the energetic path independent contour integral near crack. $J = 0$ for any close contour given that C_1 and C_2 are two arbitrary contours around the crack tip connected by segment along crack face C_3 and C_4, then a closed contour is formed. The total J along the contour is equal to the sum of contribution from each segment:

$$J = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} = 0$$

$$C_3 = C_4 = 0 \quad and \quad C_1 = -C_2$$

Therefore, any anticlockwise path that surrounds a crack will have same value of integral which is path independent. The integral is zero over a closed path, Kanninen and Popelar (1985)

III.   GLOBAL ENERGETIC PATH CONTOUR INTEGRAL

Anderson (2014), Moran and Shih (1987) investigated on the general J – integral where the simplified energy integral is modified. However, global representation of J- integral account for dynamic effect, time dependent material behavour including the inertial effect.

$$J = \frac{F}{v}$$

F = energy flux

$$J = \lim_{c=0} \int_c (w+T)\, dy - \sigma_{ij} n_j \frac{\partial u_i}{\partial x}\, ds \tag{21}$$

$w =$ stress work

$$w = \int_0^{\varepsilon_{ij}} \sigma_{ij}\, d\varepsilon_{ij}$$

$T =$ kinetic energy density

$$T = \frac{1}{2}\rho \frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t}$$

$$\partial(\sigma_{ij} u_i) = \rho \frac{\partial^2 u_{ij}}{\partial t^2} + \sigma_{ij}\frac{\partial u_i}{\partial x_j}$$

$$= \dot{T} + \dot{w}$$

$$\int_{\partial v} \sigma_{ij} u_{ij} m_j\, ds = \frac{d}{dt}\int_v (W+T)\, dv - \int_{\partial v}(w+T) v_j m_j\, ds \tag{22}$$

$v =$ volume

$m_j =$ outward normal to surface ∂v and

$v_i =$ instantaneous velocity of ∂v

In a case of two dimensional flawed solid materials where the partial separation is along *x*- axis and the origin at crack tip. Define a contour to fixed in space containing advancing crack and bounded by the area A while the crack tip is enclosed by a small contour *c* fixed in size and move with the crack. Therefore, the energy balance law is given as:

$$\int \sigma_{ij} u_i m_j\, dc = \frac{d}{dt}\iint_A (w+T)dA - \int_C \int \left[(w+T)v\partial_{ij} + \sigma_{ij} u_{ij}\right] m_j\, dc \tag{23}$$

$\int_{c_0} \sigma_{ij} u_i m_j dc =$ the rate at which energy is input into the body

$\frac{d}{dt}\int (w+T)dA =$ the rate of increase in energy of the body

The last term is the rate at which energy is lost from body due to flux through *c*

$n_j = -m_j$ on *c*

$$\therefore F(c) = \int_c \left[(w+T)v\delta_{ij} + \sigma_{ij} u_i\right] n_j\, dc$$

The flux does not depend on shape of *c:* therefore, flux to crack tip is given as $F = \lim_{c\to 0}\int_c [(w+T)v\,\delta_{ij} + \sigma_{ij} u_i]n_j\, dc$

$$\tag{24}$$

As time increases by dt, fissure extends by $dl = Vdt$ and energy expended

is fdt therefore, energy release rate is given as

$$J = \frac{F}{v} \tag{25}$$

substituting equation (24) into equation (25) give rise to a global expression for J-integral

$$\therefore u_i = -v\frac{\partial u_i}{\partial x} + \frac{\partial u_i}{\partial t} \tag{26}$$

Hence, in steady state situation, the second term in equation (26) vanishes and the displacement at fixed distance from propagating crack tip remains constant, but close to crack tip displacement changes rapidly with positions (at a fixed time). The first term is dominative in all cases in equation (26)

$$J = \lim_{c\to 0}\int_c \left[(w+T)\delta_{ij} - \sigma_{ij}\frac{\partial u_i}{\partial x} \right] n_j dc$$

$$= \lim_{c\to 0}\int_c \left[(w+T)dy - \sigma_{ij}n_j\frac{\partial u_i}{\partial x} \right] n_j dc \tag{27}$$

The above equation applies to all categories of material response: elastic, plastic, visco-plastic viscoelastic behavior. It was derived from globalize energy balance.

Equivalent Domain Global Energy Integral

Reza (2014) and Nguyen (2014) also supported the generalized integral

$$J = \lim_{c\to 0}\int_c \left[(w+T)\delta_{ij} - \sigma_{ij}\frac{\partial u_i}{\partial x} \right] n_j dc$$

$$T = \frac{1}{2}\rho\frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t} \qquad\qquad J = \int_{c^*}\left(\sigma_{ij}\frac{\partial u_i}{\partial x_j} - w\delta_{ij} \right)qm_i dc - \int_{c^+ +c^-}\sigma_{2j}\frac{\partial u_i}{\partial x_1}qdc$$

$$\sigma_{ij} = \varepsilon_{ij}^{total} = \varepsilon_{ij}^e + \varepsilon_{ij}^p + \alpha\Theta\delta_{ij} \qquad J = \int_{A^*}\frac{\partial}{\partial x_i}\left(\left(\sigma_{ij}\frac{\partial u_i}{\partial x_1} - w\delta_{ij} \right)q \right)dA - \int_{c^+ +c^-}\sigma_{2j}\frac{\partial u_i}{\partial x_1}qdc$$

$$\tag{28}$$

$$J = \int_{A^*}\left(\sigma_{ij}\frac{\partial u_i}{\partial x_1} - w\delta_{ij} \right)\frac{\partial q}{\partial x_i} + \left(\sigma_{ij}\frac{\partial \varepsilon_{ij}^p}{\partial x_1} + \frac{\partial \varepsilon_{ij}^p}{\partial x_1} + \alpha\sigma_{ii}\frac{\partial\Theta}{\partial x_1} - F_i\frac{\partial u_i}{\partial x_1} \right)qdA - \int_{c^+ +c^-}\sigma_{2j}\frac{\partial u_j}{\partial x_1}qdc \tag{29}$$

$$q = \frac{1}{\Delta l}\frac{\partial x_1}{\partial x_i}$$

Θ = thermal effect,

F_i = bodyforce

p = plasticityeffect

The above general representation of J- integralapplies to all categories of material response: elastic, plastic, visco-plastic viscoelastic behavior.The global integral permits accurate analysis possible for all deformation fields including very close proximity of crack tip.

IV.   CONCLUSION

Elastic plastic fracture dynamics that can be used in material design and evaluating the integrity of structurenotably among which is J- integral, a parameter that deals with energy released rate while stress intensity factor is a local parameter that deals with displacement and stress field in the vicinity of a fissure. The integral quantifies the magnitude of plastic zone effect that affect actual fatigue crack progression. Therefore, elastic plastic damage dynamics model for crack progression becomes necessary. This gives rise to energy flux integral that denotes path independent energy line integral and the strength of the singular stress and strain field near the crack tip that is a measure of energy available for an increment of crack extension or the energy release rate. It is also referred as crack driving and extension force. This is computed from path independent integral in the neighbourhood of flaw tip which has the potential of the release of energy from the system per unit area extension of crack growth. J integral is useful in numerical calculation of stress intensity factors. The global energy line integral account for dynamic effect, time dependent material behavior including the inertial effect. It is applied to all categories of material response: elastic plastic, visco-plastic material for a better evaluation. The global integral permits accurate analysis possible for all deformation fields including very close proximity of crack tip. Fatigue crack growth analysis are developed to support economic fail safe and damage tolerance philosophies and the execution, effectuation of damage tolerance control procedures for the workings of structural systems.

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