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On the Numerical Solutions of a Wave Equation

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Abstract*—*In this paper we have obtained approximate solutions of a wave equation using previously studied method namely perturbation-iteration algorithm (PIA). The results are compared with the first and second order difference scheme solutions by absolute error. In addition, to prove the effectiveness of the method, we have presented some graphics and tables.

Keywords*—*Initial value problems, wave equation, perturbation-iteration algorithm, difference schemes, error analysis.

I. INTRODUCTION

Partial differential equations have been used to explain many phenomena in different science and engineering branches such as mathematical biology, physics, image processing, quantum mechanics, fluid flow, viscoelasticity and so on. Therefore, to understand and explain physical interpretation of the problems arise in the above-mentioned fields, a considerable effort has been achieved and numerous methods have been proposed to obtain both numerical and analytical solutions of the partial differential equations. These methods include, Adomian decomposition method (ADM)[1-2], variational iteration method (VIM)[3-4], homotopy analysis method (HAM)[5-6], homotopy perturbation method (HPM)[7-8], finite difference method (FDM)[9-11], differential transform method (DTM)[12-13], etc.

Alongside these methods, a perturbation-iteration method, namely perturbation-iteration algorithm (PIA) has been proposed by Aksoy and Pakdemirli in 2010 [14-15]. In this paper we implementPIA to obtain some approximate solutions of a wave partial differential equation with initial conditions. Obtained results are compared with the known exact solutions and the solutions obtained by the finite difference method via first and second order difference schemes. The findings are satisfactory and the present method produces highly approximate results even for a few iterations.

II. BASIC IDEA OF PIA

In this section we introduce some fundamental points of the PIA.

Take the wave partial differential equation:

$$F(u_{tt}, u_t, uu_{xx}, u, \varepsilon) = 0 \quad (1)$$

where $u = u(x,t)$ and ε is a small perturbation parameter that will be inserted to the equation later. The perturbation expansion with only one correction term is

$$u_{n+1} = u_n + \varepsilon\,(u_c)_n \quad (2)$$

Replacing Eq.(2) into Eq.(1) and writing in the Taylor series expansion with first order derivatives only gives

$$\begin{aligned} &F((u_n)_{tt}, (u_n)_t, (u_n)_{xx}, u_n, 0)\\ &+ F_{u_{tt}}((u_n)_{tt}, (u_n)_t, (u_n)_{xx}, u_n, 0)\varepsilon((u_c)_{tt})_n\\ &+ F_{u_t}((u_n)_{tt}, (u_n)_t, (u_n)_{xx}, u_n, 0)\varepsilon((u_c)_t)_n\\ &+ F_{u_{xx}}((u_n)_{tt}, (u_n)_t, (u_n)_{xx}, u_n, 0)\varepsilon((u_c)_{xx})_n\\ &+ F_u((u_n)_{tt}, (u_n)_t, (u_n)_{xx}, u_n, 0)\varepsilon(u_c)_n\\ &+ F_\varepsilon((u_n)_{tt}, (u_n)_t, (u_n)_{xx}, u_n, 0)\varepsilon = 0 \end{aligned} \quad (3)$$

or shortly,

$$\frac{F}{\varepsilon} + ((u_c)_{tt})_n F_{u_{tt}} + ((u_c)_t)_n F_{u_t} + ((u_c)_{xx})_n F_{u_{xx}} + (u_c)_n F_u + F_\varepsilon = 0 \quad (4)$$

where $F_{u_{tt}} = \frac{\partial F}{\partial u_{tt}}, F_{u_{tt}} = \frac{\partial F}{\partial u_{tt}}, F_{u_{tt}} = \frac{\partial F}{\partial u_{tt}}, F_{u_{tt}} = \frac{\partial F}{\partial u_{tt}}$ and $F_{u_{tt}} = \frac{\partial F}{\partial u_{tt}}$.

In the expansion, allof the derivatives are calculated at $\varepsilon = 0$. Opening with the initial assumption $u_0(x,t)$, in the first step $(u_c)_0(x,t)$ is determined from Eq.(3) and then subrogated into Eq.(2) to obtain$u_0(x,t)$.The iteration procedure continues until a desired solution is obtained.

III. NUMERICAL RESULTS

Consider the following wave equation [16]

$$u_{tt} - (x+t)u_{xx} = \left(\frac{6t^4 + 4t^2 - 2}{(1+t^2)^4} + \frac{x+t}{1+t^2}\right)\sin(x) \quad (5)$$

for $0 < t < 1, 0 < x < \pi$ given with the initial and boundary conditions

$$u(x,0) = \sin(x)\,, u'(x,0) = 0, 0 \le x \le \pi$$
$$u(0,t) = u(\pi,t) = 0, 0 \le t \le 1.$$

The known exact solution of the problem is

$$u(x,t) = \frac{1}{1+t^2}\sin(x)\,. \quad (6)$$

Introducing the artificial perturbation parameter ε and rewriting Eq.(4) yields the following iteration equation.

$$\varepsilon(u_c)_{tt}(x,t) = \frac{\left(-2 + x + t\left(1 + t(6 + 2t + t^3 + (2+t^2)x)\right)\right)\sin(x)}{(1+t^2)^3} - (u_n)_{tt}(x,t) + (t+x)(u_n)_{xx}(x,t), n = 0,1,2,\dots \quad (7)$$

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Appropriate to the initial conditions, an initial estimation to begin the iteration process is proposed as $u_0(x,t) = \sin(x)$. Subrogating this initial condition in Eq.(4) and solving it gives

$$\begin{aligned}(u_c(x,t))_0 = {} & \frac{6-6t-7t^3-t^5-3t^2x-3t^4x}{6(1+t^2)}\sin(x) \\ & + \frac{6(1+t^2)(1+tx)\tan^{-1}(t)}{6(1+t^2)}\sin(x) \\ & + \frac{3(1+t^2)(t-x)\ln(1+t^2)}{6(1+t^2)}\sin(x) \\ & + c_1(x) + tc_2(x) \qquad (8)\end{aligned}$$

So the first iteration result using the initial conditions is

$$u_1(x,t) = u_0(x,t) + \varepsilon((u_c(x,t))_0) \qquad (9)$$

or

$$\begin{aligned}u_1(x,t) = \sin(x) & + \frac{6t^2+(t+t^3)(6+t^2+3tx)}{6(1+t^2)}\sin(x) \\ & - 3\frac{(1+t^2)\big(2(1+tx)tan^{-1}(t)\big)}{6(1+t^2)}\sin(x) \\ & - 3\frac{(t-x)\ln(1+t^2)}{6(1+t^2)}\sin(x) \qquad (10)\end{aligned}$$

If the procedure continues similarly, we get the following results

$$\begin{aligned}& u_2(x,t) \\ & = -\sin(x) + \frac{6t^2+(t+t^3)(6+t^2+3tx)}{6(1+t^2)}\sin(x) \\ & - \frac{3(1+t^2)(2(1+tx)\tan^{-1}(t)+(t-x)\ln(1+t^2))}{6(1+t^2)}\sin(x) \\ & + \frac{1}{720(1+t^2)}\Big(4(1 \\ & + t^2)\cos(x)\,(t(-30-20t^2+9t^4-150tx+15t^3x) \\ & + 30(t+t^3-x+3t^2x)\ln(1+t^2)) \\ & - (-1440+2520t^3+4t^8-180t(-12+x)+24t^7x \\ & + 12t^5(30+17x)-30t^2(-3-36x+10x^2) \\ & + t^6(99+30x^2)-5t^4(-37-216x+54x^2) \\ & - 30(1+t^2)(-1+t^4+t(36-8x)-36x+2x^2 \\ & - 6t^2x^2)\ln(1+t^2))\sin(x) \\ & + 60(1+t^2)\tan^{-1}(t)\big(-2(-1+t^4-6tx+2t^3x)\cos(x) \\ & + \big(36-3x+6t^2x+t^4x+t(2+36x-6x^2) \\ & + 2t^3(1+x^2)\big)\sin(x)\big)\Big) \qquad (11)\end{aligned}$$

$$\begin{aligned}u_3(x,t) = {} & \frac{1}{3175200(1+t^2)} \times -21(t+t^3)(8400(-6+x) \\ & + t(t(-33600+t(-15350+3t(5040 \\ & + 608t+105t^3))) + 20t(-1778 \\ & + 3t(420-399t+22t^3))x + 630(54 \\ & - 77t^2+2t^4)x^2 - 180(27 \\ & + 1400x)))\cos(x) + 3175200\sin(x) + (1 \\ & + t^2)(1260\cos(x)(2(10t^7+70(-6+x) \\ & - 210t^4(-2+x) - 420t^2x + 56t^6x \\ & + 7t(-7+45(-8+x)x) + 63t^5(1+x^2) \\ & - 70t^3(2+3x(-4+3x)))\tan^{-1}(t) \\ & - (-17-840x+7(2t^6+6t(20-7x) \\ & + 30t^5x - 90t^2(-4+x)x + 9x^2 \\ & + 20t^3(6+x) + 5t^4(5+9x^2)))\ln(1 \\ & + t^2)) + (t(245t^8+2205t^7x \\ & + 1260(-47+42(-30+x)x) \\ & + 45t^6(-155+126x^2) + 63t^5(560 \\ & + 313x+70x^3) - 63t^4(-9262 \\ & + 21x(-160+37x)) - 210t^2(326 \\ & + 21x(-360+67x)) + 630t(1260 \\ & + x(-1304+21x(-200+9x))) \\ & - 105t^3(-7980+x(-8966+21x(-120 \\ & + 77x)))) - 1260(-47+5t^7x+42(-30 \\ & + x)x - 315t^2(-8+x)x + 35t^4(19 \\ & + 12x) + 21t^5x(-2+x^2) + 7t^6(-10 \\ & + 3x^2) + 7t(120+x(-104+15(-24 \\ & + x)x)) + 35t^3(24+x(35-6(-4 \\ & + x)x)))\tan^{-1} - 630(-420+5t^7+7t^6x \\ & + x(152-21(-40+x)x) + 105t^2x(-13 \\ & + 2(-12+x)x) - 63t^5(-6+x^2) \\ & - 35t^3(13+6x^2) + 35t^4(12+14x \\ & - 3x^3) + 7t(-20+3x(-160+9x)))\ln(1 \\ & + t^2))\sin(x)) \qquad (12)\end{aligned}$$

In this study, the Eq.(5) in [16] is solved by using PIA and the following first-order difference scheme

$$\begin{cases} \tau^{-2}(u_{k+1}-2u_k+u_{k-1}) + A_k u_{k+1} = f_k, \\ A_k = A(t_k), f_k = f(t_k), t_k = k\tau, \\ 1 \le k \le N-1, N\tau = T, \\ \tau^{-1}(u_1-u_0) + iA_1^{1/2}u_1 = iA_0^{1/2}u_0 + \psi, u_0 = \varphi. \end{cases}$$

studied in [17] and the second-order difference scheme

$$\begin{cases} \frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} + A_k u_k + \frac{\tau^2}{4}A_k^2 u_{k+1} = f_k, \\ A_k = A(t_k), f_k = f(t_k), t_k = k\tau, 1 \le k \le N-1, N\tau = T, \\ (I+\tau^2 A_0)\tau^{-1}(u_1-u_0) \\ = \frac{\tau}{2}(f_0 - A_0u_0) + \psi, f_0 = f(0), u_0 = \varphi. \end{cases}$$

studied in [16]. The results are compared and discussed.

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IV. FIGURES AND TABLES

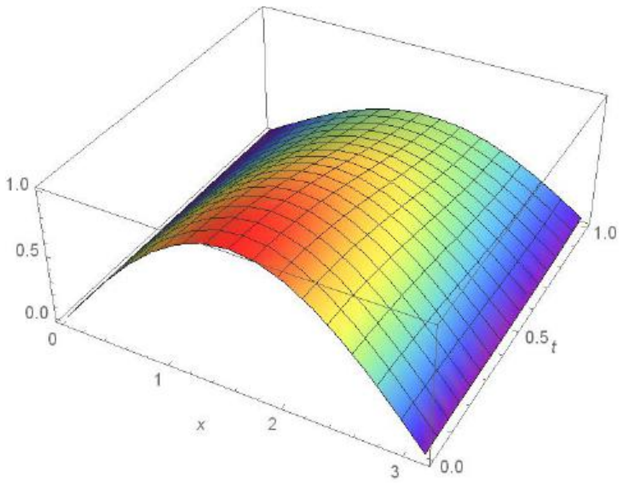

Fig. 1: Surface plot of the third order PIA solution.

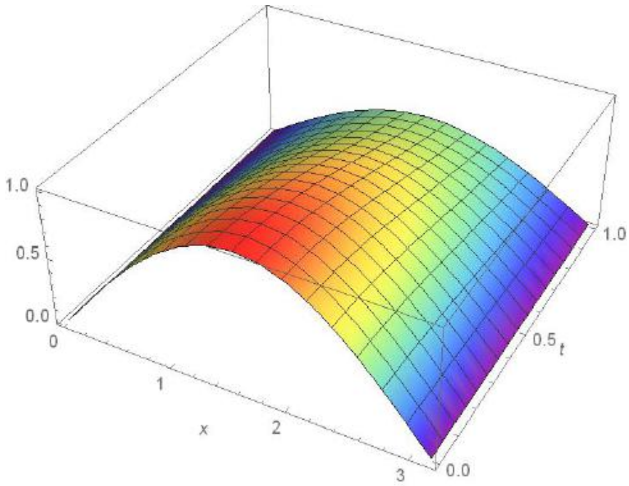

Fig. 2: Surface plot of the exact solution.

Table. 1: Comparison of the third order PIA absolute errors

	Third order PIA Absolute Errors		
x	$t = 0.2$	$t = 0.4$	$t = 0.6$
$2\pi/10$	$3.26329E-7$	$2.32795E-5$	$2.85770E-4$
$4\pi/10$	$3.23876E-7$	$2.47931E-5$	$3.24057E-4$
$6\pi/10$	$1.78211E-64$	$1.23072E-4$	$1.48412E-3$

Table. 2: Comparison of the first order difference scheme absolute errors

	First Order Difference Scheme Absolute Errors		
x	$t = 0.2$	$t = 0.4$	$t = 0.6$
$2\pi/10$	$2.66413E-4$	$4.85849E-4$	$6.71907E-4$
$4\pi/10$	$4.33106E-4$	$8.00497E-4$	$1.12590E-3$
$6\pi/10$	$4.35134E-4$	$8.14527E-4$	$1.16251E-3$

Table. 3: Comparison of the second order difference scheme absolute errors.

	Second Order Difference Scheme		
x	$t = 0.2$	$t = 0.4$	$t = 0.6$
$2\pi/10$	$6.50953E-8$	$1.19193E-7$	$4.00733E-8$
$4\pi/10$	$1.11504E-7$	$2.51531E-7$	$1.43050E-7$
$6\pi/10$	$1.71319E-7$	$5.05788E-7$	$7.22128E-7$

As shown in Table 1 and 3, the second order difference scheme is approximately 10^{-3} times better than the first order difference scheme. On the other hand, the results obtained by PIA are better than the first-order difference method but they are not as satisfactory as the results obtained second-order difference method.

For more steps of PIA, various partial differential equation scan be studied and solved and the results are compared with each other as future problems

V. CONCLUSION

In this paper the approximate solution of a wave partial differential equation is obtained by previously developed efficient method, perturbation-iteration algorithm. The method gives highly approximate solutions after a few iterations. The results are compared with the exact solution via absolute error and finite difference method. For this purpose first and second order difference schemes are applied. Also some surface plots and tables are presented to show the reliability of the method. This confirms that the method is ready to apply for wider class of partial differential equations.

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