An Application of Distributional Two Dimensional Fourier-Mellin Transform

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Abstract— Integral transforms are linear continuous operators with their inverses, transforming a class of functions to another class of functions or sequences. They provide powerful operational methods for solving initial value problems and initial-boundary value problems for linear differential and integral equations. With ever greater demand for mathematical methods to provide both theory and applications for science and engineering, the utility and interest of integral transforms seems more clearly established than ever. In spite of the fact that integral transforms have many mathematical and physical applications, their use is still predominant in advanced study and research.

Keeping these features in mind in this paper we provide the solution of differential equation for the distributional two dimensional integral Fourier-Mellin transform of the type

\[ P(\Lambda_{t,l,x,y}) f(t, l, x, y) = \int f(t, l, x, y) dt \]

and

\[ P(D_{t,l,x,y}) f(t, l, x, y) = f(t, l, x, y) \]

using the differential operator \(\Lambda_{t,l,x,y}\) and \(D_{t,l,x,y}\).

Keywords— Adjoint operator, Fourier Transform, Generalized function, Mellin Transform, Two Dimensional Fourier-Mellin Transform.

1. INTRODUCTION

Characteristically, one uses the integral transformation as mathematical or physical tool to alter the problems into one that can be solved. Integral transforms are linear continuous operators with their inverses, transforming a class of functions to another class of functions or sequences. The concept of integral transform originated from Fourier integral formula. It is composed of four objects & defined as

\[ F(s) = f(t) \mathcal{K}(t, s) dt \]

There are two functions \((F(s), f(t))\) as output, \(f(t)\) as input) that changes depending on problem and two objects which are remain same for a transform irrespective of input function \((K(t, s)\) as kernel, interval). Kernels are different for different type of transforms. They operate on function in much the same way as (certain) matrices do on vectors [2]. Integral Transform methods providing unifying mathematical approach to the study of electrical, network, devices for energy conversion and control, antennas and other component of electrical system [3]. Integral transforms transform a class of differential equations into a class of algebraic equations; the algebraic equations can be solved easily. So, we can easily find the solutions of those differential equations, this is the most useful significance of integral transforms. These are also providing powerful operational methods for solving initial value problems and initial-boundary value problems for linear differential and integral equations [1]. These are equally applied to the subject of electrical communication by wire or optical fibers, to wireless radio propagation [3]. This is a key point because it provides a way to understand exactly what an integral transform does. The integral transforms provide both theory and applications for science and engineering. Integral Transform methods are at the heart of engineering curriculum. In spite all that things their uses are still predominant in advanced study and research. So, these integral transforms are attracting to all researchers, since, their fabulous work and applications in almost all fields. In the present paper, we have focused on applications of Distributional Two Dimensional Fourier-Mellin Transform to differential equations. For this purpose, we have putting the following definitions which are already defined in our previous papers [4,7,8,9] as-

1.1. The Test Function space \(FM_{a,b,c,d,a}\)

Let \(I\) be the open set in \(R_+ \times R_+\) and \(E_+\) denotes the class of infinitely differentiable function defined on \(I\), the space \(FM_{a,b,c,d,a}\) is given by

\[ FM_{a,b,c,d,a} = \{ f: f E_+ /  \mathcal{K}_{a,b,c,d,a;rr,gg,j} \} \]

\[ \sup_{\alpha}(e^{k(t, l, x, y)}D_1^{r+a}D_2^{g}f(t, l, x, y)) < C \Delta^{k} \]

(1.1.1)

Where, the constant \(C,A\) depending on testing function space.\(^6\)

1.2. Two Dimensional Distributional Fourier-Mellin Transform of Generalized function in \(FM_{a,b,c,d,a}\)

The Two Dimensional Distributional Fourier-Mellin Transform is defined as-

\[ FM(f(t, l, x, y)) = F(s, u, p, v) \]

\[ = \langle f(t, l, x, y), e^{-t(s+u)}(x^{p-1}y^{q-1}) \rangle \]

(1.2.1)
Now we construct an operator $\Lambda^*_k$, its adjoint operator $\Lambda^*_k$. Using this differential operator in the given paper we have presented an application of Two Dimensional Fourier-Mellin Transform to differential equations. The notation and terminology are given as per A.H. Zemanian [5, 6].

II. AN APPLICATION OF DISTRIBUTIONAL TWO DIMENSIONAL FOURIER-MELLIN TRANSFORM

The kernel of Distribution Two Dimensional Fourier-Mellin Transform

$$K(t, x, y, s, u, p, v) = e^{i(st+ui)}e^{-1y^{-1}}$$

Now differentiate above term w.r.t. $t, l, x$ and $y$, we get

$$\frac{d}{dt}D_t\frac{d}{dx}D_xK(t, l, x, y, s, u, p, v) = (-is)e^{-i(tu)}u^{-1}e^{-iu}(v - 1)$$

$$\frac{d}{dx}D_x\frac{d}{dy}D_yK(t, l, x, y, s, u, p, v) = -su(p - 1)(v - 1)$$

Now multiplying both side by $xy$, we get

$$xyD_tD_xD_yK(t, l, x, y, s, u, p, v) = -su(p - 1)(v - 1)$$

Now we construct an operator

$$\Lambda^*_k \equiv xyD_tD_xD_yK(t, l, x, y, s, u, p, v) = -su(p - 1)(v - 1)$$

For each $k = 1, 2, 3, \ldots \ldots$ we can easily get $\Lambda^*_k f(t, l, x, y) = \Lambda^*_k f(t, l, x, y)$

For all $f \in \mathcal{F}^k$, we have

$$\Lambda^*_k f(t, l, x, y, s, u, p, v) = (C_0) K(t, l, x, y, s, u, p, v)$$

$$\equiv (-2su(p - 1)(v - 1))$$

Since, the operator $\Lambda^*_k K(t, l, x, y, s, u, p, v)$ is obviously linear and continuous, we have

$$\mathcal{F}[\Lambda^*_k f(t, l, x, y, s, u, p, v)] = (\Lambda^*_k \mathcal{F}f(t, l, x, y), K(t, l, x, y, s, u, p, v))$$

$$\mathcal{F}[\Lambda^*_k (t, l, x, y, s, u, p, v)] \equiv \Lambda^*_k \mathcal{F}f(t, l, x, y) = (\Lambda^*_k f(t, l, x, y, s, u, p, v))$$

$$= -2su(p - 1)(v - 1)K(t, l, x, y, s, u, p, v)$$

For all $f \in \mathcal{F}^k$.

III. ADJOINT OPERATOR

We have defined an operator $\Lambda^*_k f(t, l, x, y) = \Lambda^*_k f(t, l, x, y)$

Using the relation

$$(\Lambda^*_k f(t, l, x, y), \mathcal{F}(t, l, x, y)) = (f(t, l, x, y), \Lambda^*_k \mathcal{F}(t, l, x, y))$$

For all $f \in \mathcal{F}^k$.

We have

$$(\Lambda^*_k f(t, l, x, y), K(t, l, x, y, s, u, p, v)) = (f(t, l, x, y), \Lambda^*_k \mathcal{F}(t, l, x, y))$$

$$\equiv (f(t, l, x, y), (\Lambda^*_k f(t, l, x, y), K(t, l, x, y, s, u, p, v)))$$

For all $f \in \mathcal{F}^k$.

IV. AN APPLICATION OF THE TWO DIMENSIONAL FOURIER-MELLIN TRANSFORM TO DIFFERENTIAL EQUATION

4.1 SOLUTION OF $P(\Lambda^*_k f(t, l, x, y) = f(t, l, x, y)$

Consider the Differential equation as

$$P(\Lambda^*_k f(t, l, x, y)) = f(t, l, x, y)$$

Where, $f \in FM^k$ and $P$ is any polynomial of degree $m$. Suppose that the equation (4.1.1) possesses the solution $u$. Applying Two Dimensional Fourier-Mellin Transform to (4.1.1) we get

$$FM[P(\Lambda^*_k f(t, l, x, y)] = FM[f(t, l, x, y)]$$

Since by using

$$FM(\Lambda^*_k f(t, l, x, y)) = [-2su(p - 1)(v - 1)K(t, l, x, y, s, u, p, v)]$$
Under the assumption that the polynomial \( P \) is such that for 
\[ P(C_0) \neq 0 \]
If we further assume that the polynomial \( P \) is such that for
\[ \epsilon > 0 \]
For \( s > 0, u > 0, p > 0, v > 0 \)
Then under this assumption (4.1.2) gives-
\[ FM(u) = \frac{P(-2su(p-1)(v-1))}{P(-2st(p-1)(v-1))} \]
Applying the inversion of Two Dimensional Fourier-Mellin Transform we get
\[ u = \frac{FM(f)}{P(C_0)} \]  
(4.1.3)

We have,
\[ FM(u) = \frac{P(-2su(p-1)(v-1))}{P(-2st(p-1)(v-1))} \]
Applying the inversion of Two Dimensional Fourier-Mellin Transform to (4.2.1) we get
\[ u = \frac{FM(f)}{P(C_0)} \]  
(4.1.4)
When
\[ f \in FM_{a,b,c,d,\alpha} \]
and
\[ P(D_{l,x,y}) = \frac{P(-2su(p-1)(v-1))}{P(-2st(p-1)(v-1))} \]
We have,
\[ FM(u) = \frac{P(-2su(p-1)(v-1))}{P(-2st(p-1)(v-1))} \]
(4.1.5)

4.2 SOLUTION OF DIFFERENTIAL EQUATION

Consider the differential equation
\[ P(D_{l,x,y})u(t, l, x, y) = f(t, l, x, y) \]  
(4.2.1)
and using we get
\[ xy D_l^2 D_x^2 D_y^2 K(t, l, x, y, s, u, p, v) = \]
\[ (-is)^q(-iu)^p(p-1)!(v-1)\theta e^{-i(st+ul)x^{p-1}y^{v-1}} = \]
\[ (-is)^q(-iu)^p(p-1)^q(v-1)^qK(t, l, x, y, s, u, p, v) \]
\[ xy D_l^2 D_x^2 D_y^2 K(t, l, x, y, s, u, p, v) = \]
\[ (-is)^q(\theta i)^q(p-1)^q(v-1)^qK(t, l, x, y, s, u, p, v) \]
We have,
\[ FM(u) = \frac{FM(f)}{P(D_{l,x,y})} \]  
(4.2.2)

We can reform them to the Two Dimensional Fourier-Mellin Transform and hence we get
\[ P[x y D_l^2 D_x^2 D_y^2 K(t, l, x, y, s, u, p, v)] FM(u) = FM(f) \]  
(4.2.3)

Under the assumption that the polynomial \( P \) is such that
\[ P[x y D_l^2 D_x^2 D_y^2 K(t, l, x, y, s, u, p, v)] < \epsilon \]
for \( \epsilon > 0 \in R^n \)
Using (4.2.3)
\[ FM(u) = \frac{FM(f)}{P[x y D_l^2 D_x^2 D_y^2 K(t, l, x, y, s, u, p, v)]} \]
Applying the inversion of Two Dimensional Fourier-Mellin Transform to above equation, we have
\[ u = \frac{FM(f)}{P[x y D_l^2 D_x^2 D_y^2 K(t, l, x, y, s, u, p, v)]} \]

V. CONCLUSION

In this paper we have given an application of Distributional Two Dimensional Fourier-Mellin transform to differential equation by introducing new operator of \( \Lambda_{l,x,y} \) and its adjoint operator \( \Lambda_{l,x,y}*^{-1} \).

REFERENCES