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Functional Product of Graphs: Properties and applications

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ABSTRACT. *This paper presents a generalization of the cartesian product of graphs, which we call the functional product of graphs. We prove some properties of this new product, and we show that it is commutative, associative under certain conditions, and it has a neutral element, which consists of a single vertex without edges (the trivial graph). We present a characterization of the graphs, which can be obtained from functional product of other graphs. We prove that the maximum degree of the product graph is the sum of the maximum degrees of the factor graphs, and we present conditions that ensure the connectedness of the product graph. Finally, we present an application of the functional product of graphs, in which we prove some results that allow to generate graphs that admit an equitable total coloring, with at most $\Delta + 2$ colors.*

Keywords*: Functional product of graphs; Properties; Equitable Total Coloring.*

1 Introduction

The cartesian product of graphs was first defined by Sabidussi [14] and Vizing [18] in the 1960's. Since then, a lot of work has been done on various topics related to the product graph. The product graphs have numerous applications in diverse areas, such as Mathematics, Computer Science, Chemistry and Biology [6]. Furthermore, the cartesian product presents some important algebraic properties. These properties were investigated, independently, by Sabidussi [14] and Vizing [18]. They showed that if we identify isomorphic graphs, then the cartesian product is commutative, associative, and it has a neutral element, which consists of a single vertex without edges (trivial graph).They also demonstrated that each connected finite graph has a decomposition into prime factors that is unique except for isomorphisms. Later, several works were done studying the behavior of the cartesian product with respect to the invariants of graphs. [6, 1, 8, 13, 17].

The total coloring was introduced independently by Behzad [2] and Vizing [19], and both conjectured that every graph admits a *total coloring* with at most $\Delta + 2$ colors. The total coloring of cartesian product of graphs has been investigated by different authors [6, 12, 15, 16, 22, 23]. In [6], Kemnitz and Marangio investigated the *total chromatic number* of cartesian product of complete graphs, cycles, complete graphs and bipartite graphs, and cycles and bipartite graphs. In [15, 16], the *total chromatic number* of the cartesian product of two paths, a path and a cycle, a path and a star, a cycle and a star, and two cycles are determined. Some partial results on the total coloring of cartesian products of several paths and several cycles are contained in [22]. In [23], Zmazek and Zerovnik generalized the result on [12], determining an upper bound for the *total chromatic number* of a graph.

Recently, Lozano *et al.* [9] have studied some relationships between equitable total coloring and range vertex coloring in some regular graphs. They proved that if a regular graph admits a 2-distant coloring with $\Delta + 1$ colors, then the coloring of the vertices can be completed to an equitable total coloring with at most $\Delta + 2$ colors. In [7] Lozano *et al.* showed the equivalence of a range coloring of order Δ and the two-distant coloring [3]. These results motivated us to study the possibility of constructing families of regular graphs that admit a 2-distant coloring with $\Delta + 1$ colors.

In the section 3 of this paper, we introduce the concept of the functional product of graphs, and we show that it is a generalization of the cartesian product of graphs, and we prove some properties. In Section 4, we present an application of the functional product of graphs, and we prove some results that describe a method for obtaining harmonic graphs. We are going to show that all harmonic graphs admit an equitable total coloring with at most $\Delta + 2$ colors (i.e. it satisfies Wang's conjecture [20]). In this text, the graphs are simple, not oriented and without loops.

2 Basic Definitions and Notations

Below, we list the notations to be used throughout this paper:

- $\{u, v\}$ or uv denotes an edge of the graph G, in which u and v are adjacent;
- $d_G(v)$ or $d(v)$, if there is no ambiguity, denotes the degree of a vertex v in the graph G;
- $\Delta(G)$ or Δ, if there is no ambiguity, denotes the maximum degree of the graph G;
- $N_G(v)$ or $N(v)$, if there is no ambiguity, denotes the set of all adjacent vertices to a vertex v in the graph G;
- $F(X)$ denotes the set of all bijections of X in X;
- $D(G)$ denotes the digraph obtained by replacing each edge uv of the graph G by arcs (u, v) and (v, u) while maintaining the same set of vertices;
- $\mathcal{D}$ denotes the set of digraphs that satisfy the following conditions:
    1. (u, v) is an arc of the digraph if and only if (v, u) is also an arc of this digraph.
    2. No two arcs are alike.
- $\overrightarrow{G} \in D, G(\overrightarrow{G})$ denotes the graph obtained by replacing each pair of arcs (u, v) and (v, u) of $\overrightarrow{G}$ for the edge uv while maintaining the same set of vertices;
- $E(X)$ or E, if there is no ambiguity, denotes the set of edges (arcs) of the graph (digraph) X;
- $V(X)$ or V, if there is no ambiguity, denotes the set of vertices of the graph (digraph) X;

Definition 2.1. *[4] Let $G(V, E)$ be a graph, $S \subset (E \cup V)$ be a set, k be a natural number, and $C = \{c_1, c_2, ..., c_k\}$ be an arbitrary set whose elements are called colors. A coloring of the graph G with the colors of C is an application $f : S \to C$.*

In the above definition, if $S = V$ then f is a **vertex coloring**. In the case that $S = E$, this is called an **edge coloring**. Finally, if $S = (E \cup V)$, then f is called a **total coloring**. If $x \in S$ and $f(x) = c_i$, for $i \in \{1, 2, ..., k\}$, then we say that x owns or is colored with the color c_i.

Definition 2.2. *[4] Let $G(V, E)$ be a graph, $S \subset (E \cup V)$ be a set, and $C = \{c_1, c_2, ..., c_k\}$ be a set of colors, in which k is a natural number. A coloring $f : S \to C$ with colors from C is called a **proper coloring** if for every pair $x, y \in S$ with x adjacent or incident to y, $f(x) \neq f(y)$.*

From now on, every coloring considered in this paper is going to be proper and surjective unless it is explicitly stated otherwise.

Definition 2.3. *[21] Let $G(V, E)$ be a graph, $S \subset (E \cup V)$ be a set, and $C = \{c_1, c_2, ..., c_k\}$ be a set of colors, in which k is a natural number. A coloring $f : S \to C$ of the graph G with colors from C is called an **equitable coloring** if for every pair $i, j \in \{1, 2, ..., k\}$ we have that $||f^{-1}(c_i)| - |f^{-1}(c_j)|| \leq 1$, in which $|f^{-1}(c_i)|$ and $|f^{-1}(c_j)|$ are the cardinalities of the sets of the elements of S that have the colors c_i and c_j respectively.*

Definition 2.4. *[5] Let $G(V, E)$ be a graph and $C = \{c_1, c_2, ..., c_k\}$ be a set of colors, in which k is a natural number, an application $f : S \to C$ is called a **range vertex coloring of order** k of G. If for all $v \in V$, such that $d(v) < k$, then $|c(N(v))| = d(v)$; otherwise $|c(N(v))| \geq k$, in which $|c(N(v))|$ is the cardinality of the set of colors used in the neighborhood of v.*

Observe that range coloring generalizes some known vertex colorings. The usual vertex coloring of G is a range coloring of order one. The equivalence of a range coloring of order Δ and the two-distant coloring is showing in theorem 2.1.

Definition 2.5. *[3] Let $G(V, E)$ be a graph and $C = \{c_1, c_2, ..., c_k\}$ be a set of colors, in which k is a natural number. A coloring $f : V \to C$ with colors from C is called a **two-distant coloring** if every pair of vertices with distance 1 or 2 has different colors.*

Theorem 2.1. *[7] Let $G(V, E)$ be a graph and $C = \{c_1, c_2, ..., c_k\}$ be a set of colors, in which k is a natural number. A coloring $f : V \to C$ is a range coloring of order Δ if only if f is a **two-distant coloring**.*

3 Functional Product of Graphs

The main objective of this section is to present the definition of the functional product of graphs and to prove some properties of this new product. For this purpose, it is necessary to define applications, called linking applications, that associate each edge of a factor graph with a bijection defined on the set of vertices of another. This bijection indicates the manner in which the connection of the vertices of the product graph will be performed. We are also going to show also that the cartesian product of graphs can be viewed as a particular case of the functional product, in which all edges are associated to the identity application.

Definition 3.1. *The digraphs $\overrightarrow{G_1}(V_1, E_1)$ and $\overrightarrow{G_2}(V_2, E_2)$ are functionally linked by applications $f_1 : E_1 \to F(V_2)$ and $f_2 : E_2 \to F(V_1)$ if the following hold:*

1. *For all arc $(u, v) \in E_1$, if $(v, u) \in E_1$ then $f_1((u, v)) = (f_1((v, u)))^{-1}$.*
2. *For all arc $(x, y) \in E_2$, if $(y, x) \in E_2$ then $f_2((x, y)) = (f_1((y, x)))^{-1}$.*

3. *For every pair of arcs* $(u, v) \in E_1$ *and* $(x, y) \in E_2$*, we have that* $f_2((x, y))(u) \neq v$ *or* $f_1((u, v))(x) \neq y$.

The applications f_1 *and* f_2 *are called* ***linking applications****.*

Definition 3.2. *Let* $G_1(V_1, E_1)$ *and* $G_2(V_2, E_2)$ *be graphs. If* $D(G_1)$ *and* $D(G_2)$ *are functionally linked by applications* $f_1 : E(D(G_1)) \to F(V_2)$ *and* $f_2 : E(D(G_2)) \to F(V_1)$*, then the graphs* $G_1(V_1, E_1)$ *and* $G_2(V_2, E_2)$ *are said to be* ***functionally linked*** *through the same applications.*

Definition 3.3. *Let* $\overrightarrow{G_1}(V_1, E_1)$ *and* $\overrightarrow{G_2}(V_2, E_2)$ *be* ***digraphs*** *that are functionally linked by applications* $f_1 : E_1 \to F(V_2)$ *and* $f_2 : E_2 \to F(V_1)$*. The functional product of the digraph* $\overrightarrow{G_1}$ *with the digraph* $\overrightarrow{G_2}$ *through the applications* f_1 *and* f_2*, denoted by* $(\overrightarrow{G_1}, f_1) \times (\overrightarrow{G_2}, f_2)$*, is the digraph* $G^*(V^*, E^*)$ *defined as follows:*

- $V^* = V_1 \times V_2$.
- $(u, x), (v, y) \in E^*$ *if and only if one of the following conditions is true:*
  1. $(u, v) \in E_1$ *and* $f_1((u, v))(x) = y$;
  2. $(x, y) \in E_2$ *and* $f_2((x, y))(u) = v$.

Definition 3.4. *Let* $G_1(V_1, E_1)$ *and* $G_2(V_2, E_2)$ *be graphs that are functionally linked by applications* $f_1 : E(D(G_1)) \to F(V_2)$ *and* $f_2 : E(D(G_2)) \to F(V_1)$*. The* ***functional product*** *of the graph* G_1 *by graph the* G_2*, denoted by* $(G_1, f_1) \times (G_2, f_2)$*, is the graph* $G(\overrightarrow{G^*}(V^*, E^*))$*, in which* $\overrightarrow{G^*}(V^*, E^*) = (D(G_1), f_1) \times (D(G_2), f_2)$*.*

Figure 1 refers to definitions 3.1 and 3.7. From the original graphs (G_1 and G_2), are generated the corresponding digraphs ($D(G_1)$ and $D(G_2)$) replacing each edge of the graphs by a pair of opposing arcs.

Note that the cartesian product of graphs is a particular case of the functional product of graphs defined above, in which f_1 and f_2 assign the identity to all arcs of the corresponding digraphs. Figures 4 and 5 exemplify this relation.

3.1 Properties

It is immediate from definition 3.3 that if identify isomorphic graphs, then the functional product has neutral element, which consists of a single vertex without edges (the trivial graph). The following theorem shows that the functional product is commutative.

Theorem 3.1. *[10] Let* $G_1(V_1, E_1)$ *and* $G_2(V_2, E_2)$ *be graphs that are functionally linked by applications* $f_1 : E(D(G_1)) \to F(V_2)$ *and* $f_2 : E(D(G_2)) \to F(V_1)$*, then the graphs* $G^*(V^*, E^*) = (G_1, f_1) \times (G_2, f_2))$ *and* $G^{**}(V^{**}, E^{**}) = (G_2, f_2) \times (G_1, f_1)$ *are isomorphic graphs. In this sense, the functional product is commutative.*

Proof. Let $E'_1 = E(D(G_1))$ and $E'_2 = E(D(G_2))$, we are going to prove that given two vertices $(u, x) \in V^*$ and $(v, y) \in V^*$, the edge $\{(u, x), (v, y)\} \in E^*$ if and only if the edge $\{(x, u), (y, v)\} \in E^{**}$. Applying the definition of functional product, we have that $\{(u, x), (v, y)\} \in E^*$ if and only if:

1. $(u, v) \in E'_1$ and $f_1((u, v))(x) = y$, and $(v, u) \in E'_1$ and $f_1((v, u))(y) = (f_1((u, v))^{-1}(y) = x$ or
2. $(x, y) \in E'_2$ and $f_2((x, y))(u) = v$, and $(y, x) \in E'_2$ and $f_2((y, x))(v) = (f_2((x, y))^{-1}(v) = u$.

   Furthermore, $(x, u), (y, v) \in E^{**}$ if and only if
3. $(x, y) \in E'_2$ and $f_2((x, y))(u) = v$, and $(y, x) \in E'_2$ and $f_2((y, x))(v) = (f_2((x, y))^{-1}(v) = u$; or
4. $(u, v) \in E'_1$ and $f_1((u, v))(x) = y$, and $(v, u) \in E'_1$ and $f_1((v, u))(y) = (f_1((u, v))^{-1}(y) = x$.

Because 1 is equivalent to 4 and 2 is equivalent to 3, the theorem is proven. □

The following result presents a characterization of the graphs that can be obtained from functional product of other graphs.

Definition 3.5 (Graph orientation)**.** *Given a graph* $G(V, E)$*, the digraph* $\overrightarrow{G}(V, E')$ *is a* G *orientation if it satisfies the following conditions:*

1. *For all* $uv \in E$, $(u, v) \in E'$ *or* $(v, u) \in E'$.
2. *For all* $(u, v) \in E'$, $uv \in E$.
3. *For all* $(u, v) \in E'$, $(v, u) \notin E'$.

Note that every graph has an orientation just replace each edge uv by exactly one and only one of the arcs (u, v) or (v, u).

Definition 3.6. *Let* $G_1(V_1, E_1)$ *and* $G_2(V_2, E_2)$ *be graphs,* $f_1 : E(D(G_1)) \to F(V_2)$ *and* $f_2 : E(D(G_2)) \to F(V_1)$ *linking applications.* f_1 *(respectively* f_2*) is said to be* ***constant*** *if exists an orientation* $\overrightarrow{G_1}(V_1, E'_1)$ *(respectively* $\overrightarrow{G_2}(V_2, E'_2)$*) of* G_1 *(respectively* G_2*) such that for all pair of arcs* (u, v) *and* (x, y) *in* E'_1 *(respectively* E'_2*) we have that* $f_1((u, v)) = f_1((x, y))$ *(respectively* $f_2((u, v)) = f_2((x, y))$*).*

Definition 3.7. *Let* $G(V, E)$*,* $X_1 \subset V$ *and* $X_2 \subset V$*. The sets* X_1 *and* X_2 *are called* ***matched*** *if* $|X_1| = |X_2|$*, and there is a matching* $P \subset E$*, such that every edge of* P *has an end in* X_1 *and another in* X_2*, and* P *saturates both* X_1 *and* X_2*.*

Theorem 3.2. *Let* $G(V, E)$*,* $G_1(V_1, E_1)$ *and* $G_2(V_2, E_2)$ *be simple graphs. There are linking applications* $f_1 : E(D(G_1)) \to F(V_2)$ *and* $f_2 : E(D(G_2)) \to F(V_1)$*, such that* $G = (G_1, f_1) \times (G_2, f_2)$ *if and only if exists a bijection* $a : V_1 \times V_2 \to V$ *that satisfies:*

1. *For all* $v \in V_1 \times V_2$, $d(v) = d(a(v))$.
2. *For all edge* $uv \in E_1$, *the sets* $\{a((u,x)); x \in V_2\}$ *and* $\{a((v,y)); y \in V_2\}$ *are matched. For each edge* $e \in E_1$, *we denote by* ε_e *the corresponding matching.*
3. *For all edge* $xy \in E_2$, *the sets* $\{a((u,x)); u \in V_1\}$ *and* $\{a((v,y)); v \in V_1\}$ *are matched. For each edge* $e \in E_2$, *we denote by* ε_e *the corresponding matching.*
4. $E = (\bigcup_{e\in E_1} \varepsilon_e) \bigcup (\bigcup_{e\in E_2} \varepsilon_e)$.

Proof. Let suppose that 1,2, 3and 4 are true.

Let $\overrightarrow{G_1}(V_1, E_1')$ and $\overrightarrow{G_2}(V_2, E_2')$ arbitrary orientations of G_1 and G_2 respectively, for each arc $(u,v) \in E_1'$, we defined $g_{(u,v)} : V_2 \to V_2$ by $g_{(u,v)}(x) = y$, in which y is such that $\{a((u,x)), a((v,y))\} \in \varepsilon_{uv}$. Similarly, for each arc $(x,y) \in E_2'$ we defined $h_{(x,y)} : V_1 \to V_1$ by $h_{(x,y)}(u) = v$, in which v is such that $\{a((u,x)), a((v,y))\} \in \varepsilon_{xy}$.

We defined:

$f_1 : E(D(G_1)) \to F(V_2)$ by $f_1((u,v))(x) = \begin{cases} g_{(u,v)}(x) & if\ (u,v) \in E_1' \\ g^{-1}_{(v,u)}(x) & otherwise \end{cases}$

$f_2 : E(D(G_2)) \to F(V_1)$ by $f_2((x,y))(u) = \begin{cases} h_{(x,y)}(u) & if\ (x,y) \in E_2' \\ h^{-1}_{(y,x)}(u) & otherwise \end{cases}$

Let $G^*(V^*, E^*)$ the graph defined by $V^* = V_1 \times V_2$ and E^* is such that $\{(u,x),(v,y)\} \in E^*$ if and only if one of the following conditions is satisfied:

1. $(u,v) \in V(D(G_1))$ and $f_1((u,v))(x) = y$ or
2. $(x,y) \in V(D(G_2))$ and $f_2((x,y))(u) = v$.

Initially, we are going to prove that G^* is isomorphic to G. Note that $V^* = V_1 \times V_2$.
So, we define the bijection $b : V^* \to V$ by $b(v) = a(v)$. Let $\{(u,x),(v,y)\} \in E^*$, then:

- $(u,v) \in E(D(G_1))$ and $f_1((u,v))(x) = y$, which means that $\{b((u,x)), b((v,y))\} \in \varepsilon_{uv}$, so $\{b((u,x)), b((v,y))\} \in E$ or
- $(x,y) \in E(D(G_2))$ and $f_2((x,y))(u) = v$, which means that $\{b((u,x)), b((v,y))\} \in \varepsilon_{xy}$, so $\{b((u,x)), b((v,y))\} \in E$.

On the other hand, because of 3, if $\{b((u,x)), b((v,y))\} \in E$, we have:

- $\{u,v\} \in E_1$ and $\{b((u,x)), b((v,y))\} \in \varepsilon_{uv}$, so $f_1((u,v))(x) = y$ and $(f_1((u,v)))^{-1}(y) = x$ or $f_1((v,u))(x) = y$ and $(f_1((v,u)))^{-1}(y) = x$. In both cases $\{(u,x),(v,y)\} \in E^*$; or
- $\{x,y\} \in E_2$ and $\{b((u,x)), b((v,y))\} \in \varepsilon_{xy}$, so $f_2((x,y))(u) = v$ and $(f_2((x,y)))^{-1}(v) = u$ or $f_2((y,x))(u) = v$ and $(f_2((y,x)))^{-1}(v) = u$. In both cases $\{(u,x),(v,y)\} \in E^*$.

So, G^* is isomorphic to G.

It remains to prove that the applications f_1 and f_2 are linking applications. In fact, f_1 and f_2 satisfy conditions 1 and 2 of the linking application definition because of the way that they were defined. Now, if $uv \in E_1$ and $xy \in E_2$ are such that $f_1((u,v))(x) = y$ and $f_2((x,y))(u) = v$, then the edge $\{(u,x),(v,y)\} \in E^*$ would be a double edge. It implies that G^* (and therefore G) is not simple and this fact contradicts the hypotheses of the theorem. So, the applications f_1 and f_2 are linking applications.

Let suppose now that there are linking applications $f_1 : E(D(G_1)) \to F(V_2)$ and $f_2 : E(D(G_2)) \to F(V_1)$, such that $G(V,E) = (G_1, f_1) \times (G_2, f_2)$. We take the application $a : V_1 \times V_2 \to V$ as identity. Let $uv \in E_1$, then because of definition of linking applications $\varepsilon_{uv} = \{(u,x)(v,y) \in E : y = f_1((u,v))(x)\}$ is a matching between the sets $\{a((u,x)); x \in V_2\}$ and $\{a((v,y)); y \in V_2\}$. In a similar way, we have the matching ε_{xy} for each $xy \in E_2$.

To prove that $E = (\bigcup_{e\in E_1} \varepsilon_e) \bigcup (\bigcup_{e\in E_2} \varepsilon_e)$, just note that if $\{(u,x),(v,y)\} \in \varepsilon_{uv}$ or $\{(u,x),(v,y)\} \in \varepsilon_{xy}$, then $\{(u,x),(v,y)\} \in E$, and vice versa, if $\{(u,x),(v,y)\} \in E$, then $\{u,v\} \in E_1$ and $f_1((u,v))(x) = y$, in this case, $\{(u,x),(v,y)\} \in \varepsilon_{uv}$, or $\{x,y\} \in E_2$ and $f_2((x,y)(u) = v$, in this case $\{(u,x),(v,y)\} \in \varepsilon_{xy}$. So $E = (\bigcup_{e\in E_1} \varepsilon_e) \bigcup (\bigcup_{e\in E_2} \varepsilon_e)$, which is enough to prove the theorem. □

Theorem 3.3. *Let* $G_1(V_1,E_1)$, $G_2(V_2,E_2)$, $G_3(V_3,E_3)$ *be graphs,* $f_1 : E(D(G_1)) \to F(V_2)$, $f_2 : E(D(G_1)) \to F(V_3)$, $g_1 : E(D(G_2)) \to F(V_1)$, $g_2 : E(D(G_2)) \to F(V_3)$, $h_1 : E(D(G_3)) \to F(V_1)$ *and* $h_2 : E(D(G_3)) \to F(V_2)$ *linking application between respective graphs. If* $h_3 : E(D(G_3)) \to F(V_1 \times V_2)$ *and* $t_1 : E(D((G_1,f_1) \times (G_2,g_1))) \to F(V_3)$ *are defined by:*

$$h_3((u,v))(x,y) = (h_1((u,v))(x), h_2((u,v))(y))$$

$$t_1(((u,x),(v,y))) =$$

$$\begin{cases} f_2((u,v)) & if\ \{(u,x),(v,y)\} \in \varepsilon_{uv}, \quad uv \in E_1 \\ g_2((x,y)) & if\ \{(u,x),(v,y)\} \in \varepsilon_{xy}, \quad xy \in E_2 \end{cases}$$

Then, there are linking applications $f_3 : E(D(G_1)) \to F(V_2 \times V_3)$ *and* $t_2 : E(D((G_2,g_2) \times (G_3,h_2))) \to F(V_1)$, *such that:* $((G_1,f_1) \times (G_2,g_1),t_1) \times (G_3,h_3)$ *is isomorphic* $(G_1,f_2) \times ((G_2,g_2) \times (G_3,h_2),t_2)$.

Proof. Let $G^*(V^*,E^*) = ((G_1,f_1) \times (G_2,g_1),t_1) \times (G_3,h_3)$ and $G'(V',E') = (G_2,g_2) \times (G_3,h_2)$ be graphs. We define the bijection $a : V_1 \times V' \to V^*$ such that $a((x,(y,z))) = ((x,y),z)$. We are going to show that G^*,

G_1, G' satisfy the conditions of the theorem 3.2. From now on, $n_1 = |V_1|$, $n_2 = |V_2|$, $n_3 = |V_3|$, $V_1 = \{x_1, \cdots, x_{n_1}\}$, $V_2 = \{y_1, \cdots, y_{n_2}\}$ and $V_3 = \{z_1, \cdots, z_{n_3}\}$.

Let $\{x_{i_1}, x_{i_2}\} \in E_1$, with $i_1, i_2 \in \{1, \cdots n_1\}$. Note that the sets $\{(x_{i_1}, y_j)\}$ and $\{(x_{i_2}, y_j)\}$, $j \in \{1, ..., n_2\}$ are matched in $(G_1, f_1) \times (G_2, g_1)$ because $\{x_{i_1}, x_{i_2}\} \in E_1$ and $f_1((x_{i_1}, x_{i_2}))$ is a bijection of V_2 in V_2.

Fixing $j_1 \in \{1, \cdots, n_2\}$, the edge $\{(x_{i_1}, y_{j_1}), (x_{i_2}, f_1((x_{i_1}, x_{i_2}))(y_{j_1}))\} \in E((G_1, f_1) \times (G_2, g_1))$. As $t_1(e) = f_2((x_{i_1}, x_{i_2}))$, if $e \in \varepsilon_{x_{i_1} x_{i_2}}$, then for each $k \in \{1, \cdots, n_3\}$ the edge $\{((x_{i_1}, y_{j_1}), z_k), ((x_{i_2}, f_1((x_{i_1}, x_{i_2}))(y_{j_1})), z_k)\} \in E^*$.

So, the sets $\{((x_{i_1}, y_{j_1}), z_k)\}$ and $\{((x_{i_2}, f_1((x_{i_1}, x_{i_2}))(y_{j_1})), z_k)\}$, $k \in \{1, \cdots, n_3\}$, are matched in G^*. Now, if we take $j_2 \in \{1, \cdots, n_2\}$, with $j_2 \neq j_1$, then $f_1((x_{i_1}, x_{i_2}))(y_{j_1}) \neq f_1((x_{i_1}, x_{i_2}))(y_{j_2})$ and the respective matchings have no edges in common. This shows that $\{((x_{i_1}, y_j), z_k)\}$ and $\{((x_{i_2}, y_j), z_k)\}$, with $j \in \{1, ..., n_2\}$, $k \in \{1, ..., n_3\}$ are matched in G^*.

Let $\{(y_{i_1}, z_{k_1}), (y_{i_2}, z_{k_2})\} \in E(G')$, we are going to analyze two cases:

case 1. If $(y_{j_1}, y_{j_2}) \in E(D(G_2))$ and $g_2((y_{i_1}, y_{i_2}))(z_{k_1}) = z_{k_2}$, then $\{(x_i, y_{j_1})\}$ and $\{(f_2((y_{j_1}, y_{j_2}))(x_i), y_{j_2})\}$, with $i \in \{1, \cdots, n_1\}$ are matched in $(G_1, f_1) \times (G_2, g_1)$. As $t_1(e) = g_2((y_{i_1}, y_{i_2}))$, if $e \in \varepsilon_{y_{i_1} y_{i_2}}$, then for each $i \in \{1, \cdots, n_1\}$, the edge $\{((x_i, y_{j_1}), z_{k_1}), ((x_i, y_{j_2}), z_{k_2})\} \in E^*$. Therefore, the sets $\{((x_i, y_{j_1}), z_{k_1})\}$ and $\{((x_i, y_{j_2}), z_{k_2})\}$ are matched.

case 2. If $(z_{k_1}, z_{k_2}) \in E(D(G_3))$ and $h_2((z_{k_1}, z_{k_2}))(y_{j_1}) = y_{j_2}$, just note that $h_3((z_{k_1}, z_{k_2}))(x_i, y_{j_1}) = (h_1((z_{k_1}, z_{k_2}))(x_i), h_2((z_{k_1}, z_{k_2}))(y_{j_1})) = (h_1((z_{k_1}, z_{k_2}))(x_i), y_{j_2})$ for all $i \in \{1, \cdots, n_1\}$. It establishes a matching between sets and $\{((x_i, y_{j_1}), z_{k_1})\}$ e $\{((x_i, y_{j_2}), z_{k_2}\}$.

Now, it remains to prove that every edge of G^* is in any of the matchings. In fact, if the $\{((x_{i_1}, y_{j_1}), z_{k_1}), ((x_{i_2}, y_{j_2}), z_{k_2})\} \in E^*$, then one of the conditions below is satisfied:

case 1. If $(z_{k_1}, z_{k_2}) \in E(D(G_3))$ and $h_3((z_{k_1}, z_{k_2}))(x_{i_1}, y_{j_1}) = (x_{i_2}, y_{j_2})$, then $\{((x_{i_1}, y_{j_1}), z_{k_1}), ((x_{i_2}, y_{j_2}), z_{k_2})\}$ is in the matching between $\{((x_i, y_{j_1}), z_{k_1})\}$ *and* $\{((x_i, y_{j_2}), z_{k_2})\}$, with $i \in \{1, \cdots n_1\}$.

case 2. If $((x_{i_1}, y_{j_1}), (x_{i_2}, y_{j_2})) \in E(D((G_1, f_1)) \times (G_2, f_2)))$ and $t_1(((x_{i_1}, y_{j_1}), (x_{i_2}, y_{j_2})))(z_{k_1}) = z_{k_2}$, we have 2 subcases:

subcase 1. If $(x_{i_1}, x_{i_2}) \in E(D(G_1))$ and $f_1((x_{i_1}, x_{i_2}))(y_{j_1}) = y_{j_2}$, then $\{(x_{i_1}, y_{j_1}), (x_{i_2}, y_{j_2})\} \in E((G_1, f_1)) \times (G_2, g_1))$, so $\{((x_{i_1}, y_{j_1}), z_{k_1}), ((x_{i_2}, y_{j_2}), z_{k_2})\}$ is in the matching between $\{((x_{i_1}, y_{j_1}), z_k)\}$ and $\{((x_{i_2}, y_{j_2}), z_k)\}$, with $k \in \{1, \cdots n_3\}$.

subcase 2. If $(y_{j_1}, y_{j_2}) \in E(D(G_2))$ and $g_1((y_{j_1}, y_{j_2}))(x_{i_1}) = x_{i_2}$, then $\{(x_{i_1}, y_{j_1}), (x_{i_2}, y_{j_2})\} \in E((G_1, f_1)) \times (G_2, g_1))$, so $\{((x_{i_1}, y_{j_1}), z_{k_1}), ((x_{i_2}, y_{j_2}), z_{k_2})\}$ is in the matching between $\{((x_{i_1}, y_{j_1}), z_k)\}$ and $\{((x_{i_2}, y_{j_2}), z_k)\}$, with $k \in \{1, \cdots n_3\}$, which is enough to prove the theorem. □

See that the associativity of the cartesian product of graphs [2, 14] is a consequence of the theorem 3.3 because if the bijections associated by the linking applications are always the identity, they satisfy the conditions of the theorem.

3.2 Invariants

In this section, we prove that the maximum degree of the product graph is the sum of the maximum degrees of the factor graphs, and we present conditions that ensure the connectedness of the product graph.

Theorem 3.4. *[10] Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be graphs that are functionally linked by applications $f_1 : E(D(G_1)) \to F(V_2)$ and $f_2 : E(D(G_2)) \to F(V_1)$. For every vertex (u, x) of the graph $G^*(V^*, E^*) = (G_1, f_1) \times (G_2, f_2)$, we have that*

$$d_{G^*}(u, x) = d_{G_1}(u) + d_{G_2}(x).$$

Proof. For each $(u, x) \in V^*$, we call $E_{G^*}((u, x))$ the set of edges that are incident on that vertex in the graph G^*. The application $h_i : N_{G_1}(u) \to E_{G^*}((u, x))$ is constructed as follows. Let $h_1(v) = (v, y)(u, x)$, in which $y \in V_2$ is such that $f_1((u, v))(x) = y$, with $(u, v) \in E(D(G_1))$, in which y exists because $f_1((u, v))$ is bijective. On the other hand, h_1 is injective because if $v_1, v_2 \in N_{G_1}(u)$ and $v_1 \neq v_2$, then necessarily $(v_1, y_1)(u, x) \neq (v_2, y_2)(u, x)$ for any values of y_1 and y_2. Similarly, we construct $h_2 : N_{G_2}(x) \to E_{G^*}(u, x)$. If an edge is incident in (u, x) in the graph G^*, then it has the form $(u, x)(v, y)$. Then, it exists $(u, v) \in E(D(G_1))$, such that $f_1((u, v))(x) = y$ or $(x, y) \in E(D(G_2))$ such that $f_2((x, y))(u) = v$. Due to construction h_1 and h_2, we have that $h_1(N_{G_1}(u)) \cup h_2(N_{G_2}(v)) = E_{G^*}(u, x)$. Otherwise, if $(u, x)(v, y) \in h_1(N_{G_1}(u))$ and $(u, x)(v, y) \in h_2(N_{G_2}(v))$, then there are arcs $(u, v) \in E(D(G_1))$ and $(x, y) \in E(D(G_2))$ such that $f_1((u, v))(x) = y$ and $f_2((x, y))(u) = v$. This contradicts condition 3 of the definition of linking applications, so it holds that $h_1(N_{G_1}(u)) \cap h_2(N_{G_2}(v)) = \emptyset$.

Now, we can construct the bijection as follows:

$h : (N_{G_1}(u)) \cup (N_{G_2}(x)) \to E_{G^*}(u, v)$ defined by

$$h(a) = \begin{cases} h_1(a), & \text{if} \quad a \in N_{G_1}(u), \\ h_2(a), & \text{if} \quad a \in N_{G_2}(x). \end{cases}$$

This proves the theorem. □

From the previous theorem, we immediately obtain the following corollary.

Corollary 3.4.1. *Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be graphs that are functionally linked by applications $f_1 : E(D(G_1)) \to F(V_2)$ and $f_2 : E(D(G_2)) \to F(V_1)$. Then, the graph $G^* = (G_1, f_1) \times (G_2, f_2)$ has maximum degree $\Delta(G^*) = \Delta(G_1) + \Delta(G_2)$.*

In general, the functional product of connected graphs is not necessarily connected, as it is showing in the next proposition.

Proposition 3.1. *Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be bipartite graphs with partitions such that $V_1 = V_{11} \cup V_{12}$ and $V_2 = V_{21} \cup V_{22}$, with $|V_{11}| = |V_{12}|$ and $|V_{21}| = |V_{22}|$. Let $f_1 : E(D(G_1)) \to F(V_2)$ and $f_2 : E(D(G_2)) \to F(V_1)$ the respective linking applications, such that if $f_1(e)(u) = v$, then u and v are in different partitions of G_2 and if $f_2(e)(u) = v$, then u and v are in different partitions of G_1. Then, the graph $G^*(V^*, E^*) = (G_1, f_1) \times (G_2, f_2)$ is disconnected.*

Proof. Let $V_1 = \{0, 1, 2, \ldots, n-1\}$, and $V_2 = \{0, 1, 2, \ldots, m-1\}$, for $i = 0, 1, 2, \ldots, n-1$, and $j = 0, 1, 2, \ldots, m-1$. Without loss of generality, suppose that $V_{11} = \{0, 2, 4, \ldots, n-2\}$, $V_{12} = \{1, 3, 5, \ldots, n-1\}$, $V_{21} = \{0, 2, 4, \ldots, m-2\}$, and $V_{22} = \{1, 3, 5, \ldots, m-1\}$. Let $G^*(V^*, E^*) = (G_1, f_1) \times (G_2, f_2)$ be the functional product graph.
Let's prove that the edge $\{(i, j), (i', j')\} \in E^*$ if and only if $(i + j)$ and $(i' + j')$ have the same parity. By the definition of functional product, $\{(i, j), (i', j')\} \in E^*$ if and only if one of the following conditions is true:

1. $(i, i') \in E(D(G_1))$, and $f_2(j) = j'$ or $f_2^{-1}(j) = j'$;
2. $(j, j') \in E(D(G_2))$, and $f_1(i) = i'$ or $f_1^{-1}(i) = i'$.

In case 1, we have:
If i is even and j is even, then i' is odd and j' is odd.
If i is even and j is odd, then i' is odd and j' is even.
If i is odd and j is even, then i' is even and j' is odd.
If i is odd and j is odd, then i' is even and j' is even.
In all cases, the sum has the same parity.
In case 2, it is sufficient to proceed in a similar way to achieve the desired result. So, $G^*(V^*, E^*) = (G_1, f_1) \times (G_2, f_2)$ is disconnected and G^* has 2 connected components of the same cardinality. □

The following theorem gives a condition that ensures the connectedness of a functional product graph if the factors are connected. We are going to need two new concepts, namely **centered applications** and **centroids**.

Definition 3.8. *Let $G(V, E)$ be a graph, W be an arbitrary finite set, and $f : E(D(G)) \to F(W)$ be an application, it is said that f is **centered** if it exists $x \in W$, such that $f(e)(x) = x$ for all $e \in E$. Then, x is called a **centroid** of f.*

Theorem 3.5. *Given two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ that are connected and functionally linked by applications $f_1 : E(D(G_1)) \to F(V_2)$ and $f_2 : E(D(G_2)) \to F(V_1)$, if f_1 or f_2 is a centered application, then the functional product G_1 by G_2, with respect to f_1 and f_2, is connected.*

Proof. Without loss of generality, suppose that f_2 is centered, and let $y \in E_1$ be the centroid of f_2, $G^*(V^*, E^*) = (G_1, f_1) \times (G_2, f_2)$, and $V_2 = \{u_1, u_2, \ldots, u_n\}$. Because y is the centroid and G_2 is connected, all vertices $(y, u_i) \in V^*$, such that $i \in \{1, \ldots n\}$ are in the same connected component of G^*. Now, let $(x, u_{i_0}) \in V^*$ be arbitrary, because G_1 is connected, there is a path $x x_1 x_2 \ldots x_p$, with $x_p = y$ linking x at y in G_1. Let $u_{i_1} = f_2((x, x_1))(u_{i_0}) \ \ldots \ u_{i_p} = f_2((x_{p-1}, x_p))(u_{i_{p-1}})$, then the path $(x, u_{i_0})(x_1, u_{i_1}) \ldots (x_p, u_{i_p})$ joins the vertex (x, u_{i_0}) with (y, u_{i_p}). This proves that all of the vertices of G^* are in the same connected component. Therefore, G^* is connected. □

The cartesian product of graphs is connected if and only if both factors are connected. For more details, one can refer to [2, 14]. Note that this result is a consequence of the theorem 3.5 because, in the cartesian product of graphs, the linking applications of f_1 and f_2 assign the identity to all arcs of the corresponding digraphs, ie, both are centered applications.

4 Applications Functional Product of Graphs

In this section, we present some results that show how generate harmonic graphs from any regular graph. As consequence of theorem 4.2, the total coloring of those graphs is equitable and, in consequence, it satisfies the Wang's Conjecture. In order to better understand the following results, we first state two theorems, which appears in [21] and [11] respectively.

Definition 4.1. *[11] A regular graph G is said to be **harmonic** if it admits a range coloring of order Δ (or equivalently a two-distant coloring) with $\Delta + 1$ colors.*

Theorem 4.1 (Petersen, 1891)**.** *[21] If $G(V E)$ is a $2k$-regular graph, then G is two-factorizable.*

Theorem 4.2. *[11] Let $G(V, E)$ be a regular graph and $c : V \to C = \{1, 2, 3, \ldots,$
$\Delta + 1\}$ a range coloring of order Δ of G. Then, there is a equitable total coloring of G with at most $\Delta + 2$ colors.*

Theorem 4.3. *For any regular graph G and its complement G', there are linking applications f_1 and f_2, such that $G^* = (G, f_1) \times (G', f_2)$ is a harmonic graph.*

Proof. First note that for any regular graph G, either $\Delta(G)$ or $\Delta(G')$ is even. In fact, if $\Delta(G)$ is odd, then because $n =$

$|V(G)|$ is even, $\Delta(K_n)$ is odd and $\Delta(K_n) = \Delta(G)+\Delta(G')$, it follows that $\Delta(G')$ is even. Initially suppose that $\Delta(G')$ is even, by Theorem 4.1, there is a decomposition of G' into two-factors. For each two-factors F, replace each cycle by an oriented cycle and define the application $a : V(F) \longrightarrow V(F)$, such that if $(u, v) \in E(F)$, then $a(u) = v$ and, clearly a is a bijection.
The application f_2 associates the bijection a to each arc of the cycle and it associates the inverse bijection to each reverse oriented cycle. In the graph G, the application f_1 associates the identity to all pairs of arcs associated to edges. Now, if $V(G) = v_0, v_1, v_2, ..., v_p$, then in each vertex of the form (x, v_p), we apply the color p. By construction, the resulting coloring of $G^* = (G_1, f_1) \times (G_2, f_2)$ is a coloring with range Δ and it has $\Delta + 1$ colors. If $\Delta(G')$ is odd, then $\Delta(G)$ is even and so one only needs to change the positions of G and G', in the previous reasoning, to obtain the desired result. Therefore, $G^* = (G, f_1) \times (G', f_2)$ is a harmonic graph. □

Theorem 4.4. *Let G be a regular graph and G' be its complement. If $\Delta(G')$ is even, then for any graph H such that $\Delta(G') = \Delta(H)$ there are linking applications f_1 and f_2, such that $G^* = (G, f_1) \times (H, f_2)$ is a harmonic graph.*

Proof. It is only necessary to note that both G' and H can be decomposed in the same number of two-factors and each two-factor of G' has an associated bijection of vertices of G. Let $F_1, F_2, F_3, ..., F_t$ be the two-factors of the decomposition of G', let $r_1, r_2, ..., r_t$ be the associated bijections, and let $K_1, K_2, k_3, ..., K_t$ be the two-factors of the decomposition of H, which will be replaced by oriented cycles $O_1, O_2, ..., O_t$, the application of f_2 associates the bijections r_i to each arc O_i, and r_i^{-1} to the reverse oriented cycle for all $i \in 1, 2, ..., t$. The application of f_1 associates the identity to all edges of G. Now, if $V(G) = v_1, v_2, ..., v_p$, then in each vertex of the form (x, v_p), we apply the color p. Then, by construction, the resulting coloring of $G^* = (G_1, f_1) \times (G_2, f_2)$ is a coloring with range Δ and it has $\Delta + 1$ colors. Therefore, the graph is harmonic. □

Figures 6, 7, 8, and 9 illustrate the proof of Theorem 4.3 using a 3-regular graph with eight vertices. Figures 10, 11, 12 and 13 illustrate the proof of Theorem 4.4 using two cycles, C_5 and C_3. Figure 14 shows the equitable total coloring of the harmonic graph, obtained as a consequence of theorem 4.2.

Figures 10, 11, 12 and 13 illustrate the proof of Theorem 4.4 using two cycles, C_5 and C_3. Figure 14 shows the equitable total coloring of the harmonic graph, obtained as a consequence of theorem 4.2.

5 Conclusions

This paper presented the functional product of graph, which is a generalization of the cartesian product of graphs. We show that the functional product is commutative, it has a neutral element, and associative under certain conditions. We prove a result that offers a characterization of the product graphs, ie. it shows how are graphs that can be obtained by the functional product.

We studied some invariants. Initially, we proved that the maximum degree of the product graph is the sum of the maximum degrees of the factor graphs. In relation to connectedness, we showed that the functional product of connected graphs is not necessarily connected. We proved a result that gives some conditions in which the functional product of connected graphs is disconnected. In addition, we presented a condition that ensures the connectedness of a functional product graph if the factors are connected.

On the other hand, the functional product has proved to be efficient at constructing graphs that "inherit" desirable properties from the factors as was shown in Section 4. As application of the functional product, we proved two theorems that ensure that harmonic graphs can be constructed using the functional product of graphs and any regular graphs as basis.

In future work, it will be studied the behavior of other invariants of graphs, for example chromatic number, connectivity, dominance, and diameter. In addition, it will be studied the possibility of recognizing families or subfamilies of graphs that can be obtained by the functional product. For example, the figures below 15 and 16 illustrate the **Kneser graph** $KG_{5,2}$ isomorphic to the **Petersen Graph** generated by the functional product of a P_2 and a C_5.

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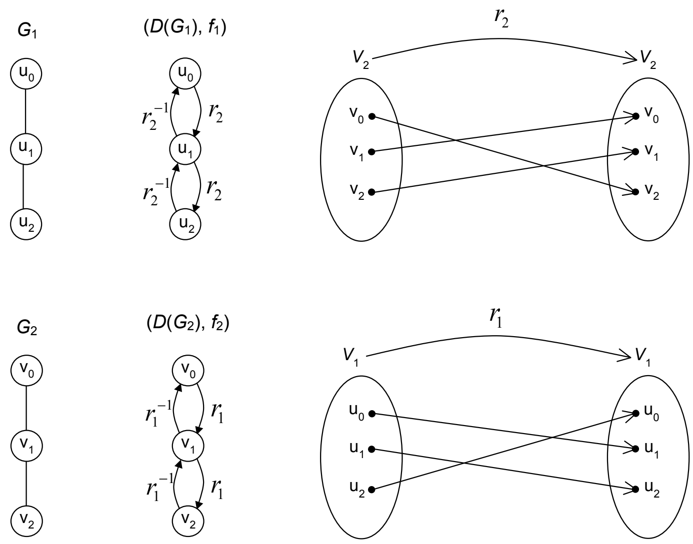

Figure 1: Graphs G_1 and G_2, respective digraphs $D(G_1)$ and $D(G_2)$, and applications f_1 and f_2.

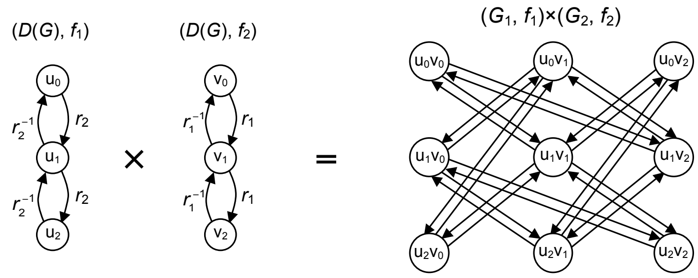

Figure 2: Functional Product between the Digraphs $D(G_1)$ and $D(G_2)$ according to f_1 and f_2.

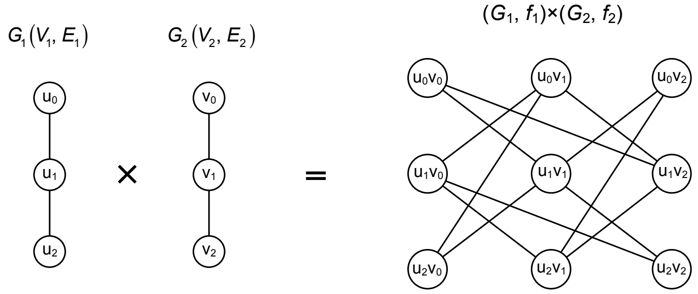

Figure 3: Functional Product between the graphs G_1 and G_2 according to f_1 and f_2.

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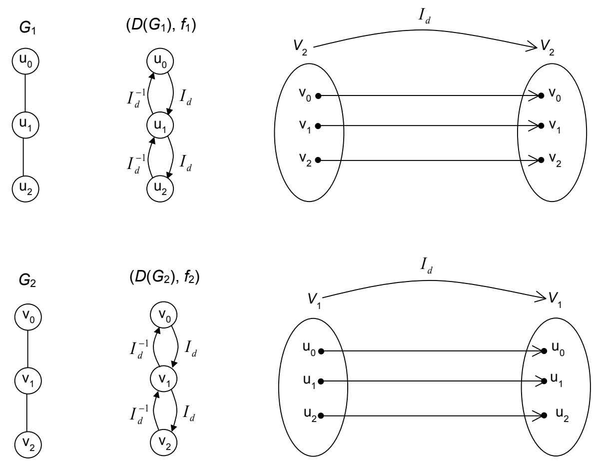

Figure 4: Graphs G_1 and G_2, their respective digraphs, and applications f_1 and f_2.

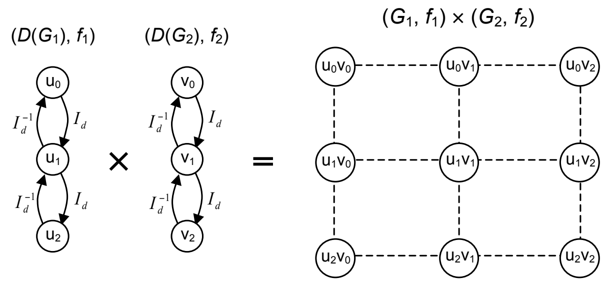

Figure 5: Functional Product (or Cartesian) between the graphs G_1 and G_2 according to f_1 and f_2.

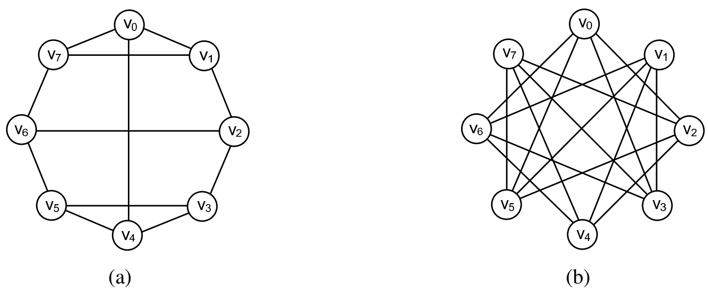

Figure 6: $3-$regular graph and its complement.

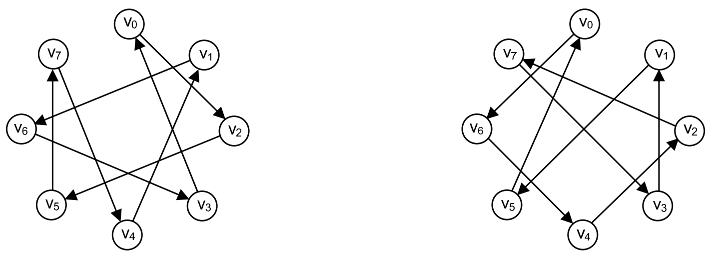

Figure 7: Cycles obtained from the decomposition of two-factors from the graph in Figure 6(b) with an arbitrary orientation.

0 → 2	0 → 6
1 → 6	1 → 5
2 → 5	2 → 7
3 → 0	3 → 1
4 → 1	4 → 2
5 → 7	5 → 0
6 → 3	6 → 4
7 → 4	7 → 3

Figure 8: Bijections associated with the cycles of Figure 7.

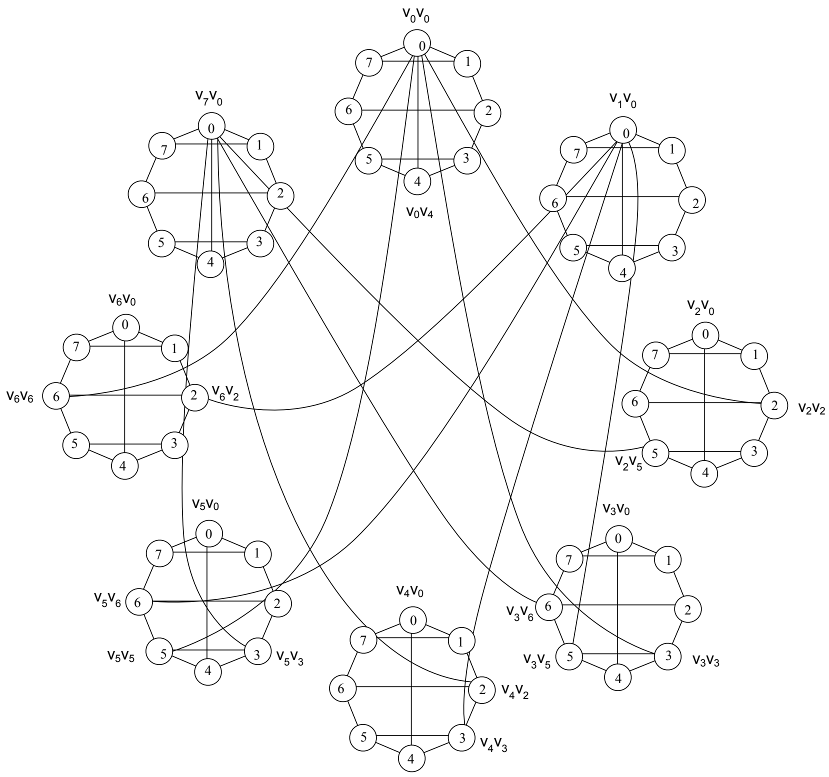

Figure 9: The start of the construction process of harmonic graphs, only the connections of the vertex $(v_0, v_0), (v_1, v_0)$ and (v_7, v_0) have been designed.

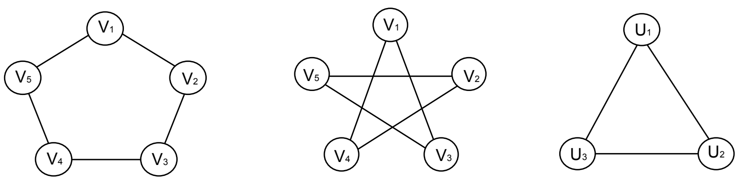

Figure 10: Graph G (C_5), its complement G', and graph H (C_3).

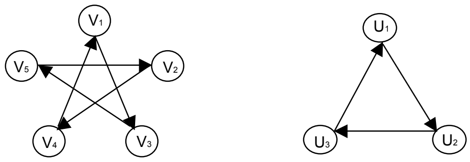

Figure 11: Cycle obtained from the decomposition of two-factors from the graph G' with an arbitrary orientation.

$1 \to 3$
$2 \to 4$
$3 \to 5$
$4 \to 1$
$5 \to 2$

Figure 12: Bijection associated with the cycle of Figure 11.

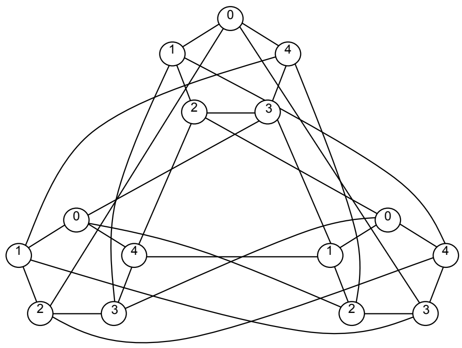

Figure 13: Range coloring of order 4 with 5 colors of the Harmonic Graph.

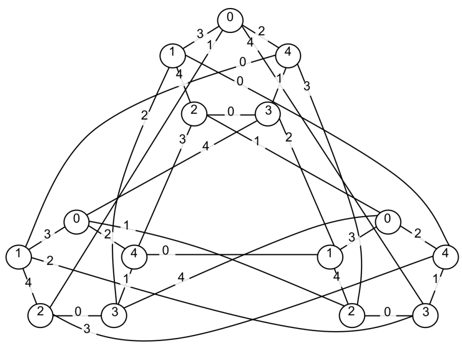

Figure 14: Equitable total coloring with 5 colors of the Harmonic Graph.

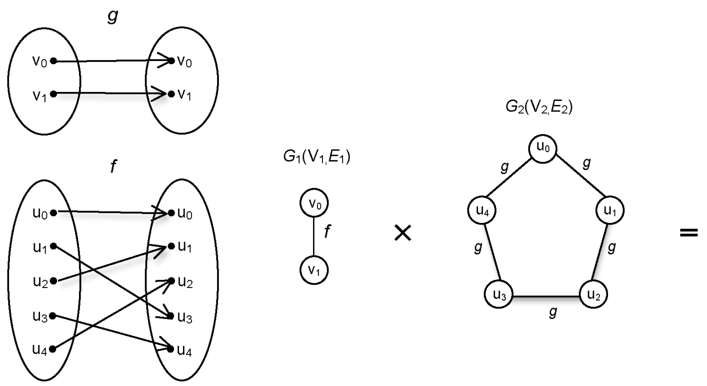

Figure 15: Graphs P_2 and C_5 with their associated bijections f and g.

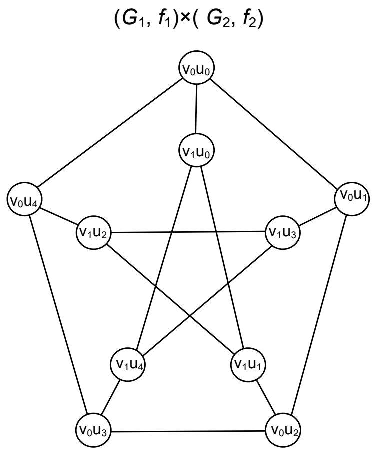

Figure 16: Petersen Graph generated by the functional product of a P_2 and a C_5 according to f_1 and f_2.