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Weighted Taylor Series for Water Wave Modeling

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Abstract*— *In this study, the Taylor series is formulated with a weighted coefficient to time step and spatial interval. With the weighted Taylor series, the weighted total acceleration is formulated on Euler's momentum equation and the Kinematic Free Surface Boundary Condition (KFSBC).

The final part is the development of a time series water wave model using the weighted momentum Euler equation and the weighted KFSBC.

I. INTRODUCTION

Hydrodynamicequationsinclude continuityequationand Euler's momentum equation formulated using the Taylor series $O(\delta^1)$ (Dean (1991). Meanwhile, KFSBC is a total velocity equation of the movement of the water surface in the direction of vertical axis that can be formulated using the Taylor series.

Analytical solutions to Laplace's equation using the separation of variables produce a sinusoidal wave equation (Dean (1991). Thus, the formulation of the equations for the water wave mechanics should be based on the nature of the sinusoidal function. Time step and spatial interval in the Taylor series for sinusoidal equations are correlated with phase speed (Courant (1928)), not with particle velocity. Hence, it is necessary to formulate a Taylor series in which time step and spatial intervals correlate with the water particle velocity. Thus, it can be used in the formulation of basic equations of hydrodynamics that are the basic equations of water wave mechanics.

The first step of this research was formulating the Taylor series for sinusoidal functions where the time step and spatial interval can be correlated with the water particle velocity. At this stage, the weighted Taylor series was produced, that is, the Taylor series in which, there is a weighted coefficient on the time step and spatial interval.

Next, with the weighted Taylor series, the basic equations of hydrodynamics were formulated. They are namely the continuity equation, the Euler's momentum equation,and KFSBC containing a weighted coefficient.

With the basic equations of hydrodynamicscontaining the weighted coefficient, the time series water wave model was formulated.

II. THE FORMULATION OF THE WEIGHTED TAYLOR SERIES

This chapter examining the meaning of $\frac{\delta x}{\delta t}$in a sinusoidal function and the meaning of $\frac{\delta z}{\delta t}$in the hyperbolic functions considering the solution of Laplace's equation which is the multiplication of a sinusoidal function with a hyperbolic function (Dean (1991). This chapter is a rewrite of Hutahaean (2021), considering that this section is the basis of the theory developed and at the same time is a correction of typos in Hutahaean (2021).

2.1. An Overview of the Solution of Laplace's equation

Solution of Laplace's equation (Dean (1991) is,

$\Phi(x,z,t) = Gcoshk(h+z)coskxsin\sigma t$.....(1)

Particle velocity in the direction of horizontal axis$-x$is,

$u = -\frac{\partial\Phi}{\partial x} = Gkcoshk(h+z)sinkxsin\sigma t$......(2)

The velocity in the direction of vertical axis-$z$is

$w = -\frac{\partial \Phi}{\partial z} = -Gksinhk(h+z)coskxsin\sigma t$...(3)

k : wave number = $\frac{2\pi}{L}$ (m^{-1})

L : wavelength (m)

σ : angular frequency = $\frac{2\pi}{T}$ (sec^{-1})

T : wave period (sec.)

h : water depth (m)

From Laplace's equation, $\frac{\delta x}{\delta t}$in the Taylor series for a sinusoidal water wave equation is not the water particle velocity, it should be the wave celerity or wave phase speed. Meanwhile $\frac{\delta z}{\delta t}$is also a function of wave celerity that is described in the following section.

2.2. A function of a single variable

The first step was examining the characteristics of δt, δxin the sinusoidal function and δzin the hyperbolic function in the Taylor series, in a function of a single variable.The formula of the Taylor series for a function with one variable is:

$$f(x+\delta x) = f(x) + \delta x\frac{df}{dx} + \frac{\delta x^2}{2!}\frac{d^2f}{dx^2} + \frac{\delta x^3}{3!}\frac{d^3f}{dx^3}$$

$$+\frac{\delta x^4}{4!}\frac{d^4f}{dx^4} + \cdots \ldots \ldots + \frac{\delta x^n}{n!}\frac{d^nf}{dx^n} \quad .....(4)$$

a. $f(t) = \cos\sigma t$

The first single-variable of sinusoidal function examinedwas f (t) = cos σt. In his function, the value of δtwas examined, in which the Taylor series can be used with only one derivative. This study was carried out using the Taylor series third order,

$$f(t+\delta t) = f(t) + \delta t\frac{df}{dt} + \frac{\delta t^2}{2!}\frac{d^2f}{dt^2} + \frac{\delta t^3}{3!}\frac{d^3f}{dt^3} \ldots (5)$$

The second and third differential terms can be ignored if the sum of the two terms is much smaller than the first term:

$$\left|\frac{\frac{\delta t^2}{2!}\frac{d^2f}{dt^2} + \frac{\delta t^3}{3!}\frac{d^3f}{dt^3}}{\delta t\frac{df}{dt}}\right| \leq \varepsilon \ldots ..(6)$$

The fourth term, fifth term, and so on can be used. However, considering that δt is a very small number, the fourth and higher differential term is a very small number that can be ignored. Equation (6) is hereinafter referred to as the optimization equation. In (6), the variable to be calculated is δt. While ε is a very small number which will determine the level of accuracy. δt in the denominator with the numerator cancel each other out,

$$\left|\frac{\frac{\delta t}{2!}\frac{d^2f}{dt^2} + \frac{\delta t^2}{3!}\frac{d^3f}{dt^3}}{\frac{df}{dt}}\right| \leq \varepsilon \quad ....(7)$$

The derivatives of the function are

$\frac{df}{dt} = -\sigma sin\sigma t$; $\frac{d^2f}{dt^2} = -\sigma^2 cos\sigma t$ dan $\frac{d^3f}{dt^3} = \sigma^3 sin\sigma t$.

The substitution of the derivative of the function in (7),

$$\left|\frac{\frac{\delta t}{2}(-\sigma^2 cos\sigma t) + \frac{\delta t^2}{6}(\sigma^3 sin\sigma t)}{-\sigma sin\sigma t}\right| \leq \varepsilon$$

This equation is valid for any value of σt as long as it is not equal to zero. It is easier to use the value of $\sigma t$where $sin\sigma t = cos\sigma t$.This is called the characteristic point. The final equation is:

$$\left|\sigma\frac{\delta t}{2} - \sigma^2\frac{\delta t^2}{6}\right| \leq \varepsilon$$

For very small δt, the term in the absolute value sign will be positive. Thus, the absolute sign can be omitted,

$$\sigma\frac{\delta t}{2} - \sigma^2\frac{\delta t^2}{6} \leq \varepsilon$$

By using an equal sign,

$$-\frac{\sigma^2}{6}\delta t^2 + \frac{\sigma}{2}\delta t - \varepsilon = 0 \quad .....(8)$$

Equation (8) is for calculating δt where the Taylor series can be used only with the first differential.

b. $f(x) = \cos kx$

Next, $\delta x$was calculated in the function$f(x) = \cos kx$. In the same way,the obtained formula is,

$$-\frac{k^2}{6}\delta x^2 + \frac{k}{2}\delta x - \varepsilon = 0 \ldots .(9)$$

Equation (9) is for calculating δx where the Taylor series can be used only with the first differential.

c. $f(z) = \cosh k(h+z)$

The function of the next variable is$f(z) = \cosh k(h+z)$. In the same way,the obtained formula is,

$$\frac{k^2}{6}\delta z^2 + \frac{k}{2}\delta z - \varepsilon = 0 \ldots ..(10)$$

With (10),δz, can be calculated, where the Taylor series can be used only with the first differential.

In Table (1), it is presented the calculation result ofδt, δx, and δz, with (8), (9), and (10), in which wave number $k$calculated using the dispersion equation of the linear wave theory, at waterdepth of h =10 m.The dispersion equation of the linear wave theory (Dean (1991) is,

$$\sigma^2 = gk\tanh kh \quad \ldots \ldots (11)$$

g :gravitational force

Table.1: The calculation results of δt, δx, and δz

T (sec.)	δt (sec.)	δx (m)	δz (m)
6	0,00191	0,01542	0,0154
7	0,00223	0,01905	0,01903
8	0,00255	0,02258	0,02255
9	0,00287	0,02603	0,026
10	0,00319	0,02942	0,02938
11	0,0035	0,03277	0,03273
12	0,00382	0,03609	0,03604
13	0,00414	0,03938	0,03933
14	0,00446	0,04265	0,04259
15	0,00478	0,04591	0,04585

With δt, δx, and δz in Table (1), $\frac{\delta x}{\delta t}$ and $\frac{\delta z}{\delta t}$ was calculated and wave celerity $C = \frac{\sigma}{k}$ was calculated. The calculation results are presented in Table (2).

Tabel.2: The value of $\frac{\delta x}{\delta t}$ and $\frac{\delta z}{\delta t}$, and wave celerity $C = \frac{\sigma}{k}$

T (sec.)	$\frac{\delta x}{\delta t}$ (m/sec)	$\frac{\delta z}{\delta t}$ (m/sec)	$C = \frac{\sigma}{k}$ (m/sec)
6	8,0677	8,05695	8,0677
7	8,54589	8,5345	8,54589
8	8,86229	8,85049	8,86229
9	9,08074	9,06864	9,08074
10	9,23739	9,22508	9,23739
11	9,35337	9,34091	9,35337
12	9,44158	9,429	9,44158
13	9,51022	9,49754	9,51022
14	9,56465	9,5519	9,56465
15	9,60854	9,59574	9,60854

It is interesting that $\frac{\delta x}{\delta t} = \frac{\delta z}{\delta t} = C$. This correlation does not only occur for dispersion equations (11). If (11) is changed, it becomes:

$$\gamma^2\sigma^2 = gk \tanh kh$$

Where γ is a positive number greater than one, wavelength resulted will be shorter and the relation of $\frac{\delta x}{\delta t} = \frac{\delta z}{\delta t} = C$ is obtained.

2.3. A function of two variables $f(x,t) = \cos kx \cos \sigma t$

The form of Taylor Series with two variables with variables (x,t), to ease the writing, it can be written:

$$f(t+\delta t, x+\delta x) = f(t,x) + s_1 + s_2 + s_3 \ldots + s_n$$

...(12)

s_1 is the first differential term, s_2 is the second differential term, and so on

Next, the optimization equation is made:

$\left|\frac{s_2+s_3}{s_1}\right| \leq \varepsilon$…(13)

Function $f(x,t) = \cos kx \cos \sigma t$, is substituted to s_1, s_2, and s_3 to (13) and made at a characteristic point where $\cos kx = \sin kx = \cos \sigma t = \sin \sigma t$. The polynomial equation for δx is:

$c_0 + c_1\delta x + c_2\delta x^2 + c_3\delta x^3 = 0$….(14)

$$c_0 = \sigma^2\frac{\delta t^2}{2} - \sigma^3\frac{\delta t^3}{6} - \sigma\delta t\varepsilon$$

$$c_1 = -\left(\sigma^2\frac{\delta t^2}{2} + \sigma\,\delta t + \varepsilon\right)k$$

$$c_2 = (1-\sigma\delta t)\frac{k^2}{2}$$

$$c_3 = \frac{k^3}{6}$$

The equation can be written into an equation for δt, However, in this study, the equation is made with input δt to calculate δx, where δt is calculated with (8). Table (3) shows the calculation results of wave number k calculated by the dispersion equation of the linear wave theory (11), with water depth of h =10 m.

Table.3 :The results for the calculation of δt and δx with (14)

T (sec.)	δt (sec.)	δx (m)
6	0,00191	0,04628
7	0,00223	0,05719
8	0,00255	0,06778
9	0,00287	0,07813
10	0,00319	0,08831
11	0,0035	0,09836
12	0,00382	0,10831
13	0,00414	0,11819
14	0,00446	0,12801
15	0,00478	0,13779

With δt and δx in Table (3), $\frac{\delta x/\delta t}{C}$ is calculated with the calculation in Table (4).

Table.4: The value of $\frac{\delta x/\delta t}{C}$

T (sec)	$\frac{\delta x}{\delta t}$ (m/sec)	$C=\frac{\sigma}{k}$ (m/sec)	$\frac{\delta x/\delta t}{C}$
6	24,2138	8,0677	3,00133
7	25,649	8,54589	3,00133
8	26,5987	8,86229	3,00133
9	27,2543	9,08074	3,00133
10	27,7245	9,23739	3,00133
11	28,0726	9,35337	3,00133
12	28,3373	9,44158	3,00133
13	28,5433	9,51022	3,00133
14	28,7067	9,56465	3,00133
15	28,8384	9,60854	3,00133

In contrast to the results of separatecalculations, using equations derived from equations $f(x,t)$ it was obtained that $\frac{\delta x}{\delta t}=3.00133\ C$, this fits the criteria of Courant (1928) that $\frac{\delta x}{\delta t}=3\ C$.

1.4. A function with threevariables $,f(x,z,t)=\cos kx\cos\sigma t\cosh k(h+z)$

The Taylor series for a function with three variables up to the third derivative is $f(t+\delta t,x+\delta x,z+\delta z)=f(t,x,z)+s_1+s_2+s_3$

with s_1, s_2, and s_3 in Table (5)

Table.5: Element s_1, s_2 *and* s_3

s_1	s_2	s_3
$\delta t\frac{\partial f}{\partial t}$	$\frac{\delta t^2}{2}\frac{\partial^2 f}{\partial t^2}$	$\frac{\delta t^3}{6}\frac{\partial^3 f}{\partial t^3}$
$+\delta x\frac{\partial f}{\partial x}$	$+\delta t\delta x\frac{\partial^2 f}{\partial t\partial x}$	$+\frac{\delta t^2}{2}\delta x\frac{\partial^3 f}{\partial t^2\partial x}$
$+\delta z\frac{\partial f}{\partial z}$	$+\delta t\delta z\frac{\partial^2 f}{\partial t\partial z}$	$+\frac{\delta t^2}{2}\delta z\frac{\partial^2 f}{\partial t^2\partial z}$
	$+\frac{\delta x^2}{2}\frac{\partial^2 f}{\partial x^2}$	$+\delta t\frac{\delta x^2}{2}\frac{\partial^3 f}{\partial t\partial x^2}$
	$+\delta x\delta z\frac{\partial^2 f}{\partial x\partial z}$	$+\delta t\delta x\delta z\frac{\partial^3 f}{\partial t\partial x\partial z}$
	$+\frac{\delta z^2}{2}\frac{\partial^2 f}{\partial z^2}$	$+\delta t\frac{\delta z^2}{2}\frac{\partial^3 f}{\partial t\partial z^2}$
		$+\frac{\delta x^3}{6}\frac{\partial^3 f}{\partial x^3}$
		$+\frac{\delta x^2}{2}\delta z\frac{\partial^3 f}{\partial x^2\partial z}$
		$+\delta x\frac{\delta z^2}{2}\frac{\partial^3 f}{\partial x\partial z^2}$

Substitution,

$f(x,z,t)=\cos kx\cos\sigma t\cosh k(h+z)$

to s_1, s_2 and s_3 and optimization equation done at characteristic points and in conditions $\cosh k(h+z)=\sinh k(h+z)$, equations for δz was obtained, where δt and δx as input, δt was calculated using (8) while δx was calculated using (14),

$c_0+c_1\delta z+c_2\delta z^2+c_3\delta z^3=0$….(15)

With elements of c_0, c_1, c_2 and c_3 in Table (6)

The condition cosh k (h + z) = sinh k (h + z) can be obtained in deep water. However, it does not mean that the obtained equation only applies to deep waters, it also applies to shallow waters. This is considering the conservation law of the wave number (Hutahaean (2020):

$\frac{\partial k(h+z)}{\partial x}=0$ ……..(16)

Table.6: Elementsof c_0, c_1, c_2, *and* c_3

c_0	c_1	c_2	c_3
$\varepsilon\sigma\delta t$	$-\varepsilon k$	$\frac{k^2}{2}$	$\frac{k^3}{6}$
$+\varepsilon k\delta x$	$-\sigma k\delta t$	$-\frac{\sigma k^2}{2}\delta t$	
$-\sigma^2\frac{\delta t^2}{2}$	$-k^2\delta x$	$-\frac{k^3}{2}\delta x$	
$+\sigma k\delta t\delta x$	$-\sigma^2 k\frac{\delta t^2}{2}$		
$-k^2\frac{\delta x^2}{2}$	$+\sigma k^2\delta t\delta x$		
$+\sigma^3\frac{\delta t^3}{6}$	$-k^3\frac{\delta x^2}{2}$		
$+\sigma^2 k\frac{\delta t^2}{2}\delta x$			
$+\sigma k^2\delta t\frac{\delta x^2}{2}$			
$+k^3\frac{\delta x^3}{6}$			

Table (7) shows the calculation resultsof δt, δx, and δz where k was calculated by the dispersion equation of the linear wave theory (11), with the water depth of h =10 m.

Table.7: The calculation results of δt, δx and δz

T	δt	δx	δz
6	0,00191	0,04628	0,13914
7	0,00223	0,05719	0,17195
8	0,00255	0,06778	0,20379
9	0,00287	0,07813	0,23491
10	0,00319	0,08831	0,26551
11	0,0035	0,09836	0,29573
12	0,00382	0,10831	0,32566
13	0,00414	0,11819	0,35536
14	0,00446	0,12801	0,38489
15	0,00478	0,13779	0,41427

With δt, δx and δz in Table (7), $\frac{\delta x}{\delta t}$ and $\frac{\delta z}{\delta t}$ was calculated with the results presented in Table (8)

Table.8: The calculation results of $\frac{\delta x}{\delta t}$ and $\frac{\delta z}{\delta t}$ and C

T (sec)	$\frac{\delta x}{\delta t}$ (m/sec)	$\frac{\delta z}{\delta t}$ (m/sec)	$C = \frac{\sigma}{k}$ (m/sec)
6	24,2138	72,8026	8,0677
7	25,649	77,1177	8,54589
8	26,5987	79,9729	8,86229
9	27,2543	81,9442	9,08074
10	27,7245	83,3578	9,23739
11	28,0726	84,4044	9,35337
12	28,3373	85,2004	9,44158
13	28,5433	85,8197	9,51022
14	28,7067	86,311	9,56465
15	28,8384	86,7071	9,60854

With $\frac{\delta x}{\delta t}$ and $\frac{\delta z}{\delta t}$ dan C in Table (8) $\frac{\delta x/\delta t}{C}$, $\frac{\delta z/\delta t}{C}$ and $\frac{\delta z}{\delta x}$ was calculated with the results presented in Table (9).

Table.9: The calculation results of $\frac{\delta x/\delta t}{C}$, $\frac{\delta z/\delta t}{C}$ and $\frac{\delta z}{\delta x}$

T (sec)	$\frac{\delta x/\delta t}{C}$	$\frac{\delta z/\delta t}{C}$	$\frac{\delta z}{\delta x}$
6	3,00133	9,02395	3,00665
7	3,00133	9,02395	3,00665
8	3,00133	9,02395	3,00665
9	3,00133	9,02395	3,00665
10	3,00133	9,02395	3,00665
11	3,00133	9,02395	3,00665
12	3,00133	9,02395	3,00665
13	3,00133	9,02395	3,00665
14	3,00133	9,02395	3,00665
15	3,00133	9,02395	3,00665

Referring to the calculation resultsin Table (9) relations can be formulated:

$\delta x = \frac{\sigma}{k}\,\gamma\,\delta t$…..(17)

$\delta z = \frac{\sigma}{k}\gamma^2\,\delta t$ …….(18)

With separate claculation as a function of single variable, $\frac{\delta x}{\delta t} = \frac{\delta z}{\delta t} = C$ was obtained, or $\gamma = 1$ whereas with the simultaneous calculation $\frac{\delta x}{\delta t} = 3C$ and $\frac{\delta z}{\delta t} = 9C$ is obtained or $\gamma = 3$, all are related to wave celerity C.Thus,to make $\frac{\delta x}{\delta t}$ closer to horizontal velocity u and $\frac{\delta z}{\delta t}$ closer to vertical velocity w, Weighted Taylor series $O(\delta^1)$on sinusoidal function $f(x,t)$should be in the form,

$f(x+\delta x, t+\delta t) = f(x,t) + \gamma\delta t\frac{\partial f}{\partial t} + \delta x\frac{\partial f}{\partial x}$.(19)

Where the total acceleration obtained,

$\frac{Df}{dt} = \gamma\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x}$ ……(20)

Meanwhile, for the function $f(x,z,t)$, the form of Taylor series $O(\delta^1)$is,

$f(x+\delta x, z+\delta z, t+\delta t) =$

$f(x,z,t) + \gamma^2\delta t\frac{\partial f}{\partial t} + \gamma\delta x\frac{\partial f}{\partial x} + \delta z\frac{\partial f}{\partial z}$…(21)

With total acceleration,

$\frac{Df}{dt} = \gamma^2\frac{\partial f}{\partial t} + \gamma u\frac{\partial f}{\partial x} + w\frac{\partial f}{\partial z}$…..(22)

III. WEIGHTED CONTINUITY EQUATION, EULER'S MOMENTUM EQUATION,AND KFSBC

3.1. Weighted Continuity Equation.

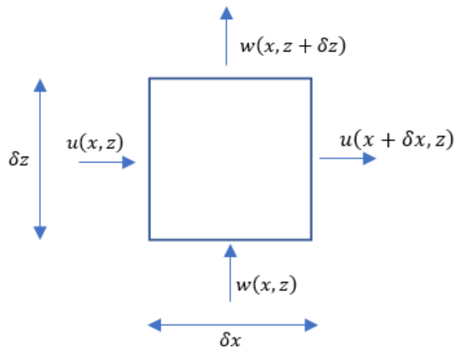

Fig.1: Control Volume for the Continuity

Equation Formulation

At $t = t$, thus (22) can be written,

a. For constant$z$

$f(x+\delta x, z, t) = f(x,z,t) + \gamma \delta x \frac{\partial f}{\partial x}$....(23)

b. For constant$x$

$f(x, z+\delta z, t) = f(x,z,t) + \delta z \frac{\partial f}{\partial z}$......(24)

The law of conservation of mass for the volume of a constant control volume (Fig. (1) and for incompressible flow,

$I - O = 0$

$I = \rho u(x,z,t)\,\delta z + \rho w(x,z,t)\,\delta x$

$$O = \rho\left(u(x,z,t) + \gamma \delta x \frac{\partial u}{\partial x}\right)\delta z + \rho\left(w(x,z,t) + \delta z \frac{\partial w}{\partial z}\right)\delta x$$

Subtraction and equation are divided by$\rho\delta x\delta z$, weighted continuity eqation is obtained,

$\gamma \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$ ....(25)

3.2.Weighted Euler's Momentum Equation

Using (22), weighted Euler's Momentum Equation in the direction of the direction of horizontal axis-$x$and in the vertical axis-$z$are

$\gamma^2 \frac{\partial u}{\partial t} + \gamma u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$ ....(26)

$\gamma^2 \frac{\partial w}{\partial t} + \gamma u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$....(27)

3.3. Weighted KFSBC.

The known KFSBC (Dean (1991) is,

$$w_\eta = \frac{\partial \eta}{\partial t} + u_\eta \frac{\partial \eta}{\partial x}$$

$w_\eta$is the water particle velocity on the surface which is the total velocity of the water level elevation$\eta(x,t) = \cos kx \cos \sigma t$, while theweighted total acceleration of water level elevation with(20) is

$$\frac{D\eta}{dt} = \gamma \frac{\partial \eta}{\partial t} + u_\eta \frac{\partial \eta}{\partial x}$$

Thus, wigthed KFSBC is,

$w_\eta = \gamma \frac{\partial \eta}{\partial t} + u_\eta \frac{\partial \eta}{\partial x}$.....(28)

IV. THE APPLICATION INTIME SERIES WATER WAVE MODELING

In this section, the governing equations for time series modeling water wavesare formulated. The governing equations consists of two equations, they are the water surface elevation equation and the particle velocity equation. The variable of particle velocity in this equation is the depth-averaged velocity.

a. Water surface elevation equation

The Continuity equation (25)is multiplied by $\partial z$and integrated with water depth. . Integration of the first term is completed with the Leibniz integral (Protter, Murray, Morrey, Charles, 1985).KFSBC and bottom boundary condition were calculated,

$\gamma \frac{\partial}{\partial x}\int_{-h}^{\eta} u\, dz - (\gamma - 1)u_\eta \frac{\partial \eta}{\partial x} + \gamma \frac{\partial \eta}{\partial t} = 0$...(29)

The integration of the left-hand first term is solved by using the particle velocity equation for the solution of Laplace's equation (2). From (2), the relation of the direction of horizontal axis of particle velocity at an elevation z to the horizontal velocity at elevation η is

$$u = \frac{\cosh k\,(h+z)}{\cosh k(h+\eta)} u_\eta$$

Left hand integration (29) becomes,

$$\int_{-h}^{\eta} u\, dx = \int_{-h}^{\eta} \frac{\cosh k\,(h+z)}{\cosh k(h+\eta)}\, dz\ u_\eta$$

Integration is completed using $\eta = \frac{A}{2}$and defined by $H = h + \frac{A}{2}$ and calculated in deep water depth wheretanh $k\,(h+\eta) = 1$,

$$\int_{-h}^{\eta} u\,dx = \frac{u_\eta}{k}$$

Conservationlaw of the wave number (Hutahaean (2020) is,

$$\frac{\partial k\left(h+\frac{A}{2}\right)}{\partial x} = 0$$

or

$$k\left(h+\frac{A}{2}\right) = k_0\left(h_0+\frac{A_0}{2}\right)$$

In deep water $\tanh k_0\left(h_0+\frac{A_0}{2}\right) = 1$where$k_0\left(h_0+\frac{A_0}{2}\right) = \theta\pi$, a relation is obtained

$$k = \frac{\theta\pi}{\left(h+\frac{A}{2}\right)} = \frac{\theta\pi}{H}$$

The final result of integration is,

$$\int_{-h}^{\eta} u\,dx = \frac{u_\eta H}{\theta\pi}$$

Substitute to (29),

$$\frac{\partial\eta}{\partial t} = -\frac{1}{\theta\pi}\frac{\partial u_\eta H}{\partial x} + \frac{(\gamma-1)}{\gamma}u_\eta\frac{\partial\eta}{\partial x} \text{.....(30)}$$

As mentioned earlier, the modeling uses depth-averaged velocity. The horizontal depth average velocity $U$is defined as the particle velocity at the leevation$z = z_0$below the SWL, where$z_0$is a negative number. From (2):

$$\frac{u_\eta}{U} = \frac{\cosh k\,H}{\cosh k(h+z_0)}$$

Is defined:

$$\alpha = \frac{\cosh kH}{\cosh k(h+z_0)} \quad \text{......(31)}$$

Thus, the relation of horizontal surface velocity with horizontaldepth-averaged velocityis:

$$u_\eta = \alpha U \qquad \text{......(32)}$$

Substitute to (30)

$$\frac{\partial\eta}{\partial t} = -\frac{\alpha}{\theta\pi}\frac{\partial UH}{\partial x} + \frac{(\gamma-1)}{\gamma}\alpha U\frac{\partial\eta}{\partial x} \text{.....(33)}$$

$z_0$is calculated by the following equation,

$$\frac{1}{UH}\int_{-h}^{A/2} u\;dz = 1$$

From (2) and the definition ofdepth-averaged velocity,

$$u = \frac{\cosh k\,(h+z)}{\cosh k(h+z_0)}U$$

The characteristic of$z_0$is,

$$\frac{1}{UH}\int_{-h}^{A/2}\frac{\cosh k\,(h+z)}{\cosh k(h+z_0)}\,dz\;U = 1$$

From this equation the equation for $z_0$is formulated:

$$kH\;coshk(h+z_0) - \sinh kH = 0 \text{...(34)}$$

The calculation ofαin (31)and in the calculation ofz_0 (34), in the deep water depth,requires deep water depth depth value h_0. wave number $k_0$is calculated bydeep-water weighted linear wave dispersion equation,

$$k_0 = \frac{\gamma^2\sigma^2}{g}$$

As deepwater depth:

$$h_0 = \frac{\theta\pi}{k_0} - \frac{A_0}{2}$$

Deep water depth is used to calculateα.Considering conservation law of the wave number, the value of $\alpha$is constant.

b. Horizontal velocity equation

Weighted horizontal surface momentum equation (Hutahaean (2021), is,

$$\gamma^2\frac{\partial u_\eta}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left(\gamma u_\eta u_\eta + w_\eta w_\eta\right) = -g\frac{\partial\eta}{\partial x}$$

By substituting surface velocity with (32) andby neglecting convective acceleration,

$$\frac{\partial U}{\partial t} = -\frac{g}{\alpha\gamma^2}\frac{\partial\eta}{\partial x} \text{....(35)}$$

c. Model Results

The Finite Difference Method for spatial differentials uses numerical solutions, while time differentials are solved by the predictor-corrector method for numerical integration (Hutahaean, 2019). The time step $\delta t$was determined with (8), using $\varepsilon = 0.005$, while the grid size of$\delta x$was calculated with (17).The model executiom was made using weighting coefficient $\gamma = 3.0$, deep water coefficient $\theta = 2.0$, $\tanh\theta\pi = 0.99999$.

As the first case, the model was done on a channel with a constant depth $h = h_0 = 14$ m. In the channel there are sinusoidal waves with wave period$T = 8$ sec. and wave amplitude $A = 1.0$ m. The model results are presented in Fig. (2). Fig (2) shows that the model can simulate well the short waves with large amplitudes.

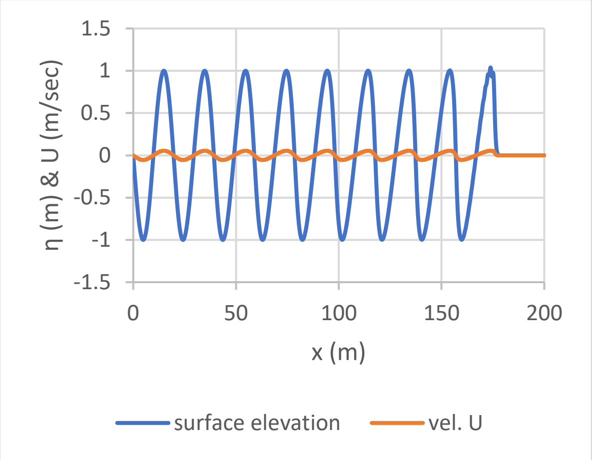

Fig.2: Model Results on Flat Bottom

In the next case, the model was made on a sloping bottom with a bottom slope $\frac{dh}{dx} = -\frac{13}{200}$. Downstream water depth is $h_0 = 14$ m, while upstream water depth is 1.0 m. The incoming wave of sinusoidal wave with the wave period of$T = 8$ sec. and wave amplitude of$A = 0.8$ m. The model results are presented in Fig.(3).

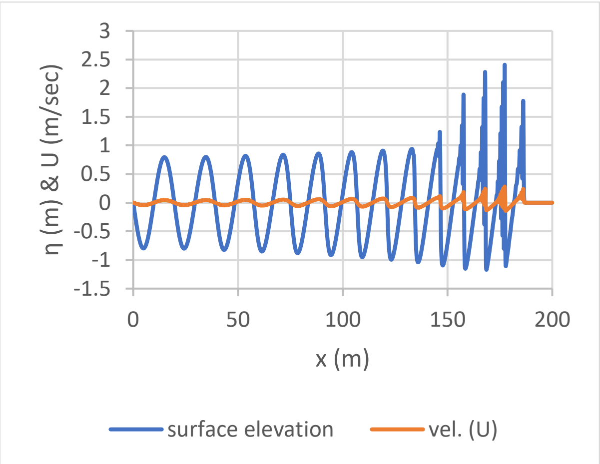

Fig.3: Model Results on Sloping Bottom.

The model results show that initially shoaling occurred, then the waves became unstable at waterdepth of 4 m and then the breaking peak occurs at a water depth of 2.60 m.

V. CONCLUSION

Some conclusions are drawn from this study. The first is that the application of the Taylor series to the sinusoidal wave equation should use time step and spatial intervals correlated with phase speed. Thus, it can be correlated with the water particle velocity, a weighting coefficient must be obtained. The Taylor series with the weighting coefficient is hereinafter referred to as the weighted Taylor series that only uses the first derivative.

The formulation of hydrodynamic equations with the weighted Taylor series produces equations with the weighting coefficient, including the weighted continuity equation, the weighted Euler's momentum equation,and the weighted KFSBC.

The next conclusion is that by using the weighted equations, the time series of the wave equation is obtained to simulate a shortwave where short wavelengths are produced and there is a breaking phenomenon.

The determination of the time step and gridsize in numerical modeling using the Finite difference method can use the equations formulated in this study.

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