Some Variants of Water Wave Dispersion Equation, Formulated with Small Amplitude Wave Assumption

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Abstract*— *This research formulates some dispersion equations with formulating procedure similar to the one in formulating dispersion equation of the small amplitude and long wave theory, i.e. by applying velocity potential equation on the Bernoulli surface equation and Kinematic Free Surface Boundary Condition equation.

Furthermore, this research uses non-linear term of the Bernoulli equation, whereas the Kinematic Free Surface Boundary Condition equation is applied with two scenarios, i.e. neglected non-linear term and not-neglected non-linear term

Wave length from various dispersion equations that are obtained are then compared with breaker length of the breaker index equation.

This research aims only to show that using similar governing equations can be obtained some dispersions equations to produce different wave length.

***Keywords*— Dispersion Equation, Wave Length.**

I. INTRODUCTION

Wave length is an important parameter of a water wave. Various phenomena in a water wave that are determined by wave length are among others shoaling and breaking, refraction and diffraction, wave force on a structure, and sediment transportation by a wave. Therefore, a dispersion equation that produces an appropriate wave length is needed.

Dean (1991) formulated dispersion equation an equation to calculate wave length, using two basic equations, i.e. Bernoulli equation at the surface and Kinematic Free Surface Boundary Condition (KFSBC) equation. In both equations, the non-linear term is neglected, by applying an assumption of small amplitude and long wave.

In this research, dispersion equation is formulated using similar governing equation, i.e. Bernoulli surface equation and KFSBC equation by keep applying the small amplitude wave assumption but without applying the long wave assumption. The non-linear term at the Bernoulli equation is still used, whereas the KFSBC is applied in two scenarios, i.e. the neglected non-linear term and the not neglected non-linear term. In the scenario where KFSBC equation is not neglected, the formulation is applied with two different approaches.

The wave length from the resulting dispersion equation is compared with breaker length calculated by breaker index equations from Komar and Gaughan (1972), Mc. Cowan (1894) and Miche (1944). Breaker height is calculated using equation from Komar and Gaughan (1972), using input breaker height breaker depth is calculated withMc.Cowan (1894) equation, using input breaker height and breaker depth, breaker length is calculated using Miche (1944) equation.

II. VELOCITY POTENTIAL EQUATION

By completing Laplace equation with variable separation method, Dean (1991) obtained the following velocity potential equation,

$$\Phi(x,z,t) = Acoskx\ (Ce^{kz} + De^{-kz})sin(\sigma t) + Bsinkx(Ce^{kz} + De^{-kz})sin(\sigma t) \quad ....(1)$$

This equation has two components, i.e. $coskx$component and $sinkx$ component. Hutahaean (2019) shows that both components have similar wave constant, where (1) can be written as,

$$\Phi(x,z,t) = A(coskx + sinkx)(Ce^{kz} + De^{-kz}) sin(\sigma t) \quad .....(2)$$

https://dx.doi.org/10.22161/ijaers.76.38 *ISSN: 2349-6495(P) | 2456-1908(O)*

The $coskx$ and $sinkx$ functions have an intersection point where the two functions have similar values, so at that point the velocity potential equation can be written as,

$$\Phi(x,z,t) = 2Acoskx\,(Ce^{kz} + De^{-kz})sin(\sigma t)$$

A new constant is defined, i.e. $A = 2A$

$$\Phi(x,z,t) = Acoskx\,(Ce^{kz} + De^{-kz})sin(\sigma t)$$
....(3)

This velocity potential (3) is an equation at the characteristic point i.e. kx point where $coskx = sinkx$. The formulation of A, C, D constants using (3) will produce a constant value at the characteristic point, where the constant value applies at all points at the wave curve.

At the formulation A, C, D constants with (3), flat bottomis used (Dean (1991), i.e.by applying the bottom water kinematic boundary condition, where at the flat bottom $\frac{dh}{dx} = 0$ applies where $h(x)$ is water depth. Hence, the bottom water kinematic boundary condition in the form of $w = -u\frac{dh}{dx}$ becomes $w = 0$, or $\frac{\partial\Phi}{\partial z} = 0$ at $z = -h$. The following is obtained

$$\Phi(x,z,t) = Gcoskx\,cosh(h+z)sin\sigma t \quad ......(4)$$

Where wave constant $G = 2ADe^{kh}$ is defined. The detail of the formulation can be seen in Dean (1991).

III. DISPERSION EQUATION OF THE LINEAR WAVE THEORY

3.1.Water Surface Equation of the Linear Wave Theory

In this section dispersion equation of linear wave theory or small amplitude and long wave theory is formulated with a procedure corresponds to the one in Dean (1991).

3.1. Applying Bernoulli Equation at the Surface,

The Bernoulli equation at the surface is,

$$-\frac{\partial\Phi_\eta}{\partial t} + \frac{1}{2}(u_\eta^2 + w_\eta^2) + g\eta + \frac{p_\eta}{\rho} = C(t) \quad ......(5)$$

By applying an assumption that the wave amplitude is very small then η, where η is the water surface elevation vis-à-vis still water level, will also be very small, so that the surface pressure can be considered as equal on the entire surface and if the reference is the atmospheric pressure then $p_\eta = 0$.

By applying an assumption that the wave amplitude is very small, then the velocity of particles u and w are also very small number so that in Bernoulli equation the second term is much smaller than the third term and therefore can be neglected.

$$-\frac{\partial\Phi_\eta}{\partial t} + g\eta = C(t)$$

Substitute the potential flow equation, water surface equation is obtained, i.e.,

$$\eta(x,t) = \frac{G\sigma}{g}coskx\,cosk(h+\eta)cos\sigma t + \frac{C(t)}{g}$$

or

$$\eta(x,t) = \frac{G\sigma}{g}coskx\,coshk\left(1+\frac{\eta}{h}\right)cos\sigma t + \frac{C(t)}{g}$$

For a very small wave amplitude A, then $\frac{\eta}{h} \ll 1$, therefore the last equation becomes,

$$\eta(x,t) = \frac{G\sigma}{g}coskx\,coshkcos\sigma t + \frac{C(t)}{g}$$

$\eta(x,t)$ has an average value against time that is equal to zero, then $C(t) = 0$, the water surface equation becomes,

$$\eta(x,t) = \frac{G\sigma coskh}{g}coskx\,cos\sigma t$$

or

$$\eta(x,t) = Acoskx\,cos\sigma t \quad ......(6)$$

It is defined that

$$A = \frac{G\sigma coskh}{g} \quad .....(7)$$

where A is the wave amplitude. (6) can be written to be an equation for G, i.e.

$$G = \frac{Ag}{\sigma coshkh} \quad .....(8)$$

$$\Phi(x,z,t) = Gcoskx\,coshk(h+z)sin\sigma t \quad ....(9)$$

(9) is velocity potential equation of the linear wave theory. The velocity particle equation in the horizontal direction at the surface for small amplitude and long wave is

$$u_\eta = -\frac{\partial\Phi}{\partial x}\bigg|_{z=\eta} = Gksinkx\,coshkhsin\sigma t$$
........(10)

Whereas the particle velocity in the vertical direction at the surface is

$$w_\eta = -\frac{\partial\Phi}{\partial z}\bigg|_{z=\eta} = -Gkcoskx\,sinhkhsin\sigma t$$
........(11)

To ease the writing, constant G is still used with the value as in (8).

3.2. Applying the Surface Kinematic Boundary Condition

The next constant to be formulated its equation is wave number k, which will be formulated using surface kinematic boundary condition equation, i.e.,

$w_\eta = \frac{\partial \eta}{\partial t} + u_\eta \frac{\partial \eta}{\partial x}$ ....(12)

For small amplitude andlong wave, $\frac{\partial \eta}{\partial x}$is a very small number and can be neglected, hence (9) becomes

$w_\eta = \frac{\partial \eta}{\partial t}$ ...(13)

Substitute (11) and (6),

$$-Gkcoskx\ sinhkhsin\sigma t = -\frac{G\sigma^2 coskh}{g} coskx\ sin\sigma t$$

A relation is obtained, i.e.

$\sigma^2 = gk\ tanhkh$ ....(14)

This equation is a dispersion equation of linear wave theory to calculate wave number k, whereas wave length can be calculated with a relation $L = \frac{2\pi}{k}$.

In deep water, the relation$tankh = 1$ applies, hence, dispersion equation becomes,

$\sigma^2 = gk$

From this equation, wave number in the deep water is obtained, i.e.

$k_0 = \frac{\sigma^2}{g}$ ....(15)

IV. DISPERSION EQUATION OF SHORT WAVE

In this section, the formulation of dispersion equation will be done using Bernoulli equation at the surface and KFSBC equation as in the formulation of dispersion equation of linear wave theory. However, the long wave assumption is not applied and a complete Bernoulli equation is used instead.

4.1. Water Surface Equation of a Complete Bernoulli Equation

In this section, water surface equation will be formulated using a complete Bernoulli equation i.e. (5). Substitute (9), (10) and (11) to (5), using$C(t) = 0$,

$$\eta(x,t) = \frac{G\sigma}{g} coskx\ coshkhcos\sigma t - \frac{G^2}{2g} k^2 sin^2 kx\ cosh^2 khsin^2 \sigma t$$

$-\frac{G^2}{2g} k^2 cos^2 kx\ sinh^2 khsin^2 \sigma t$ ...(16)

(16) is non-linear water surface equation for small amplitude wave. Time differential from (16) is,

$$\frac{\partial \eta}{\partial t} = -\frac{G\sigma^2}{g} coskx\ coshkhsin\sigma t$$
$$-\frac{G^2}{g} k^2 \sigma sin^2 kx\ cosh^2 khsin\sigma tcos\sigma t$$
$$-\frac{G^2}{g} k^2 \sigma cos^2 kx\ sinh^2 khsin\sigma tcos\sigma t$$

........(17)

4.2. Applying KFSBC Equation

As with the formulation of dispersion equation in the previous section, dispersion equation is formulated using KFSBC equation, i.e. (12). Substitute (9) and (10) to (12),

$\frac{\partial \eta}{\partial t} = -Gkcoskx\ sinhkh - Gksinkx\ coshkh \frac{\partial \eta}{\partial x}$ ....(18)

By equalizing (17) with (18) and applying it at the characteristic point, where$coskx = sinkx$and the characteristic point is also applied at domain time t, i.e.$cos\sigma t = sin\sigma t$, the following is obtained

$$-\frac{\sigma^2}{g} coshkh - \frac{G}{2g} k^2 \sigma\ (cosh^2 kh + sinh^2 kh) = -k\ sinhkh - k\ coshkh \frac{\partial \eta}{\partial x}$$

.....(19)

4.2.1. KFSBC linearand$G$ linear

In this section, the long wave assumption is applied where the second term in the right side (19) is very small compared to the first term, and relation$G$i.e. (8)isused to obtain dispersion equation in other form, i.e. ,

$\sigma^2 = gk\ tanhkh - \frac{gA}{2} k^2 (1 + tanh^2 kh)$ ...(20)

Equation (20) shows that wave number or wave length is also determined by wave amplitude.

4.2.2. KFSBC nonlinear, G linear, $\frac{\partial \eta}{\partial x}$ linear

In (19),$G$of (8) is substituted to (19) whereas $\frac{\partial \eta}{\partial x}$is substituted with (6) and was applied at the characteristic point to obtain

$\sigma^2 = gk\ tanhkh\ - \frac{gA}{2} k^2 (2 + tanh^2 kh)$ ...(21)

(21) is another form of dispersion equation, where the wave amplitude is the parameter, so that the resulting wave length is also determined by wave amplitude.

4.2.3. KFSBC nonlinear, $G$and$\frac{\partial \eta}{\partial x}$ nonlinear.

Substitute$\frac{\partial \eta}{\partial x}$to (19) with differential (16) against horizontal x axisand apply to the characteristic point , the following equation is obtained

https://dx.doi.org/10.22161/ijaers.76.38 ISSN: 2349-6495(P) | 2456-1908(O)

$$Gk^2\sigma\,(cosh^2kh + sinh^2kh) = -2\sigma^2 coshkh + 2gk\,sinhkh - \left(G\sigma k^2 cosh^2kh + \frac{G^2}{2}k^4 coskh\right) \text{.......(22)}$$

The water surface elevation $\eta$of (6) at the characteristic point is,

$$\eta = \frac{A}{2} \quad \text{......} \quad (23)$$

Then (16) is applied to the characteristic point and is equalized with (23), and the following equation is obtained,

$$gA = G\sigma\,coshkh - \frac{G^2}{4}k^2(cosh^2kh + sinh^2kh) \quad \text{......(24)}$$

(22) and (24) are two simultaneous equations with unknown wave constant $G$and wave number k. With input wave amplitude A, water depth $h$and wave period $T$where$\sigma = \frac{2\pi}{T}$, wave contant $G$and wave number $k$can be calculated with (22) and (24).

The result of wave length calculation with (14), (20), (21) and with the system of equation (22) and (24) is presented in table (1). On that table,$L_{14}$is the wave length calculated with (14), $L_{20}$is the wave length calculated with (20), and so forth

Table 1. Wave Length from four dispersion equations

h (m)	Wave length L (m)			
	L_{14}	L_{20}	L_{21}	L_{22+24}
20	88,79	83,66	80,43	80,42
19	87,63	82,57	79,36	79,39
18	86,35	81,38	78,19	78,26
17	84,96	80,07	76,88	77,01
16	83,45	78,63	75,45	75,63
15	81,79	77,04	73,87	74,11
14	79,98	75,31	72,12	72,42
13	78,01	73,4	70,19	70,56
12	75,85	71,3	68,07	68,49
11	73,49	68,98	65,71	66,21
10	70,9	66,43	63,1	63,66

Table (1), presents the result of wave length calculation with (14), (20), (21) and ((22)+(24)) using a wave with wave period $T = 8$seconds and wave amplitude $A = 1.0$ m. The result of the calculation shows that$L_{14}$is the longest, whereas, $L_{21}$and$L_{22+24}$is more or less equal although L_{21} is relatively shorter. In addition, there is a constraint at ((22)+(24)), i.e. it cannot be used in a shallow water. Henceforth, ((22)+(24)) can no longer be used.

Wave length calculation is then done with wave period $T = 8$seconds and wave amplitude $A = 1.0$ m in a shallow water, with the result of the calculation as presented in table (2), where$\delta = \frac{L_{\cdot 14} - L_{21}}{L_{\cdot 14}} x100\%$.

Table 2.Comparison of wave length in a shallow water.

h (m)	Wave length L (m)			δ (%)
	L_{14}	L_{20}	L_{21}	
20	88,79	83,66	80,43	9,42
18	86,35	81,38	78,19	9,46
16	83,45	78,63	75,45	9,58
14	79,98	75,31	72,12	9,83
12	75,85	71,3	68,07	10,26
10	70,9	66,43	63,1	11
8	64,9	60,47	56,94	12,27
6	57,5	52,98	49,08	14,65
4	48,01	43,14	38,38	20,04
2	34,69	28,5	19,34	44,24

In the deep water, the difference between L_{14}andL_{21}could reach 9.5 %, whereas in shallow water the difference could reach 44 %.

Furthermore, the effect of wave amplitude A on (21) will be studied using a wave with wave period of $T = 8$ sec., with various wave amplitudes, i.e. 0.20 m, 0.60 m and 1.0 m, with the result of the calculation as presented in table(3). It shows that the larger the wave amplitude the shorter the wave length. It can be concluded that wave amplitude is to shorten the wave length.

Table.3: Wave Length from (21) at various wave amplitude values

h (m)	Wave Length L (m)		
	$A = 0.2$ (m)	$A = 0.6$ (m)	$A = 1.0$ (m)
20	87,28	84,04	80,43

https://dx.doi.org/10.22161/ijaers.76.38 ISSN: 2349-6495(P) | 2456-1908(O)

18	84,88	81,72	78,19
16	82	78,91	75,45
14	78,56	75,52	72,12
12	74,45	71,44	68,07
10	69,5	66,49	63,1
8	63,48	60,42	56,94
6	56,02	52,8	49,08
4	46,37	42,75	38,38
2	32,6	27,66	19,34

To see the effect of the difference in wave length, particle velocity in the direction of horizontal $x$is used with the result of the calculation as presented in table (4), using a wave with wave period $T = 8$ sec., wave amplitude $A = 1.0$ m, and velocity calculated at $z = -0.25$ h. The calculation of the wave number is done using (14), (20) and (21). In table (4) t $u14$is the particle velocity calculated using wave number from (14), and so forth, whereas$\delta = \frac{u21-u14}{u14} x100$ %.

Table.4. Particle velocity in the direction of horizontal x, u

h	$u14$	$u20$	$u21$	δ
(m)	(m/sec)	(m/sec	(m/sec	(%)
20	0,66	0,68	0,69	5,66
18	0,7	0,72	0,74	6,02
16	0,74	0,77	0,79	6,47
14	0,8	0,83	0,86	7,05
12	0,87	0,91	0,94	7,84
10	0,96	1,01	1,04	8,97
8	1,08	1,14	1,19	10,76
6	1,25	1,34	1,43	13,98
4	1,55	1,7	1,88	21,55
2	2,2	2,64	3,73	69,64

The difference between$u14$ and$u21$is quite large where the shallower the water the greater the difference.

V. COMPARISON WITH BREAKERINDEXES.

As a comparator of wave length produced by dispersion equation, breaker length of breaker Indexesare used. The procedure of calculating the breaker length using breaker indexes is as follows

Breaker height that is calculated with the Komar and Gaughan equation (1972) is

$$\frac{H_b}{H_0} = 0.56\left(\frac{H_0}{L_0}\right)^{-1/5} \quad ....(24)$$

$H_b$is breaker height, $H_0$is deep water wave height andL_0 is deep water wave length calculated with (15).

Breaker depth that is calculated with McCowan (1894) equation is

$$\frac{H_b}{h_b} = 0.78 \quad ......(25)$$

$h_b$is breaker depth, whereas breaker height $H_b$is obtained from (24).

Breaker length that is calculated using Miche (1944) equation is

$$\frac{H_b}{L_b} = 0.142 tanh\left(\frac{2\pi h_b}{L_b}\right) \quad ....(26)$$

$L_b$is breaker length. Breaker height $H_b$was obtained from (24) whereas breaker depth $h_b$was obtained from (25).

The calculation of breaker height with (24) requires an input of deep water wave height $H_0$and wave period $T$for the calculation of deep water wave length L_0. Those two parameters were obtained by applyingWiegel equation (1949,1964).

By establishing a wave period T, deep water wave height $H_0$is calculated using Wiegel equation (1949,1964), i.e.

$$T = 15.6\sqrt{\frac{H_0}{g}} \quad .....(27)$$

or

$$H_0 = \frac{gT^2}{15.6^2} \quad ......(28)$$

Where$g$is gravitational velocity $\left(9.81 \ ^m/_{sec^2}\right)$, deepwater wave height $H_0$is in meter unit.The result of the calculation of the breaker length with this procedure is presented in table (5).

Table.5: Breaker length $L_b$in various wave periods.

T	H_0	H_b	h_b	L_b
(sec.)	(m)	(m)	(m)	(m)
6	1,45	1,69	2,16	20,41
7	1,98	2,3	2,95	27,78
8	2,58	3	3,85	36,28
9	3,27	3,8	4,87	45,92
10	4,03	4,69	6,01	56,69

11	4,88	5,68	7,28	68,6
12	5,8	6,75	8,66	81,63

Furthermore, with breaker depth $h_b$and with an assumption of sinusoidal wave where$A = \frac{H_b}{2}$, wave length is calculated with (14), (20) and (21), with the result as presented in table (6), where on the table,$L_{14}$is wave length of (14), $L_{20}$is wave length of (20) and$L_{21}$is wave length of(21).

Table.6: The comparison of wavelength of dispersion equation with breaker length L_b

T (sec.)	L_b (m)	L_{14} (m)	L_{20} (m)	L_{21} (m)
6	20,41	26,53	22,05	15,67
7	27,78	36,11	30,02	21,33
8	36,28	47,17	39,21	27,86
9	45,92	59,7	49,62	35,26
10	56,69	73,7	61,26	43,54
11	68,6	89,18	74,13	52,68
12	81,63	106,13	88,22	62,7

Table (6) shows that $L_{20}$is the closest toL_b. However, if it is viewed based on wavesteepness criteria of Michell (1893) where$\frac{H}{L} = 0.142$, then the one that makes wave steepness to be closer to critical wave steepness isL_{21}, as presented in table (7).

Table.7: Comparison of wave steepness

T (sec.)	$\frac{H_b}{L_b}$	$\frac{H_b}{L_{14}}$	$\frac{H_b}{L_{20}}$	$\frac{H_b}{L_{21}}$
6	0,083	0,064	0,077	0,108
7	0,083	0,064	0,077	0,108
8	0,083	0,064	0,077	0,108
9	0,083	0,064	0,077	0,108
10	0,083	0,064	0,077	0,108
11	0,083	0,064	0,077	0,108
12	0,083	0,064	0,077	0,108

VI. CONCLUSION

As a conclusion, from a governing equation can be obtained e some dispersion equations that produce various wave lengths. The higher the level of the precision, the shorter the wave length. Even using an assumption of small amplitude wave, dispersion equation with wave amplitude as its parameter can be resulted. The influence of wave amplitude is to shorten the wave length.

Variety of dispersion equations producing variety of wave lengths require a criteria on the appropriate wave length. One of the criteria that can be used is critical wave steepness,

Further research needed is formulating dispersion without applying an assumption of small amplitude wave and by taking into account the criteria of critical wave steepness.

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