https://dx.doi.org/10.22161/ijaers.611.45 ISSN: 2349-6495(P) | 2456-1908(O)

Golden ratio, Concentric Circumferences and Planetary Distances

Sebastião Martins Siqueira Cordeiro[2], Antonio Maia de Jesus Chaves Neto[3], Reginaldo Barros[4], José Francisco da Silva Costa*[1]

[1]University Campus of Abaetetuba- Faculty of Formation and Development of the Field-FADECAM- Manoel de Abreu-Abaetetuba Pará.
[2,4]University Campus of Abaetetuba - Faculty of Exact and Natural Sciences - Rua Manoel de Abreu-Abaetetuba Pará.
[3] Post graduationProgram of Natural ResourcesEngineering of Amazon– PRODERNA; ITEC, StateUniversityof Pará, 2626, 66.050-540, Belém, PA, Brazil.
*corresponding author: jfsc@ufpa.br

Abstract— *In this work we will show some important relations that includes the study of the golden ratio between two concentric circles, showing that there is a linear combination between the radii of the circumferences. We derive a constant K which corresponds to the perfect number representing the largest root of the golden ratio which is a function of the radii of two concentric circles* C_1*and*C_2*, respectively. This relation makes it possible to find the values of the radii* r_1*of* C_1*and* r_2*of* C_2 *or vice versa. We will apply the results obtained in a problem related to the uranium and Neptune planets where we will use the known astronomical distances of said planets with respect to the sun to calculate the minimum and maximum distance comparing the percentage of the relative error with the known astronomical values in the scientific literature. Plot the graphs comparing the planetary distances in relation to the distances of Kepler and Titus as well as the margin of error.*
Keywords— ***Golden Ratio; Concentric Circumferences; Planetary Distances.***

I. INTRODUCTION

The Golden Ratio is known as the numerical pattern that governs the balance of bodies along with the harmony of forms and motions in nature. From the mathematical understanding of this proportion it is possible to verify it in several phenomena of nature [1], [2].

Due to this characteristic, it is known that in several areas of knowledge, studies seek to unravel the mysteries that relate the golden ratio, also known as Fibonacci sequence [3], with the behavior of the most varied natural phenomena. Such studies generally have as purpose to analyze and to lead the equations characteristic of different phenomena, to an irrational number, which is denominated "gold number", represented by the letter $\Phi = (1 + \sqrt{5}) / 2 = 1.618034$ ... [4], [5]. Therefore, given that nature presents itself obeying a certain harmony [6], and that the existence of golden proportion is intrinsic in nature in different forms or design [7].

Due to the importance and study directed in this area, the present article tries to show that there is a relation of the golden proportion with the planetary distances, starting from the development of equations originating from two concentric circles that are positioned from this proportion.

Many texts focus on the golden ratio in the solar system and in the universe. There is the presence of this proportion in the diameters of the Earth and the Moon and they determine a triangle whose dimensions are related to the number Phi (root from the equation that represents a golden segment) and relations of that number with the distances of the planets with respect to the sun and that exponentially correlate with Phi. This number is also related to the rings of Saturn in close and dimensioned values at the golden ratio of the planet's diameter [7].

Due to this study, the present article seeks to show that it is possible to obtain planetary distances with small margins of errors by considering an equation relating the planetary rays to the sun, starting with the priore, to consider two concentric circles intersected by points whose distances are in a golden ratio. Based on this equation, they determine the smallest and largest distances of the planets Uranus and Neptune with small margins of error. We extend the equation to determine the distances of the other planets and compare them with the astronomical distances of Kepler and Titus-Bode and the error margins to detect to what extent they approach the known astronomical values [8].

Another fact that has been a reason for great admiration are the celestial bodies that fascinated the human mind since the earliest times [9], [10]. Due to scientific advances on the solar system, mathematical formalism on the movements of planets and stars as well as knowledge of their structures from astronomical observations and

https://dx.doi.org/10.22161/ijaers.611.45 ISSN: 2349-6495(P) | 2456-1908(O)

mathematical calculations [11] has led man to an understanding beyond knowing about the gravitational forces, nature of orbits, velocities and periods of revolutions relative to the sun.

Development on the solar system carried out in history by renowned scientists like Kepler, Nicolaus Copernicus, Galileo and Newton, led mankind in understanding the planetary system, with the sun in focus and the planets orbiting, elliptically, in planes and governed by gravitational forces. Another well-known scientist who sought to determine the planetary distances was Titus-Bode who was able, by a very simple law, to determine planetary distances without, however, resorting to any equation such as Kepler's laws or the law of universal gravitation [12].

Titus-Bode who was director of the Berlin Observatory, which ended up defining the final sequence, which today is known as Titus-Bode Law. This law is based on a geometric progression of reason 2, from the second term: 0, 1, 2, 4, 8, 16 and 32. Titus-Bode multiplied each of these terms by 3: Obtained: 0 , 3, 6, 12, 24, 48 and 96 and added 4 units each, yielding: 4, 7, 10, 16, 28, 52 and 100 and finally dividing by 10, obtained the following result , 0.4, 0.7, 1.0, 1.6, 2.8, 5.2 and 10.0 This sequence of numbers gave the distances of the planets to the sun [13]

II. THE PERFECTNUMBER OF NATURE: THE DIVINEPROPORTION

There is a number in nature that has been the subject of great research since antiquity and has always aroused the curiosity and fascination of mathematicians and scholars. This number that will be the target of this article corresponds to the number that we denominate of Φ = (1 + √5) / 2 = 1.618034 ... also denominated gold number. In human proportion, in works of classical architecture, Renaissance paintings and sculptures, and in nature there is a relation between the proportions of these elements and the number Φ, and for this and another reason it is considered as a magic number that organizes the universe into a same proportion known as the divine proportion. [14].

2.1 Definition of golden ratio

Called "golden proportion" by Euclid (360-295 BC) and "divine proportion" by Kepler, it was found that in the works of Leonardo da Vinci (1452-1519) such a proportion was adopted in important works. Thus, the golden ratio represents the most harmonious form of dividing into two parts of a segment so that from this division we derive the following quadratic equation.

$$k^2 - k - 1 = 0. \qquad (1)$$

Mathematically, the golden ratio can be described as follows: Let a segment of line AB be divided by a point C between A and B, the golden ratio occurs when the relationship between the sequences is satisfied,

$$\mathrm{AB}/\mathrm{AC} = \mathrm{AC}/\mathrm{CB} \qquad (2)$$

Geometrically, this relationship can be visualized as shown in Figure 1.

Fig.1: Golden ratio of a line segment AB divided by a point C.

Solving the proportion (1), we have to

$$(\mathrm{AC})^2 = \mathrm{AB} \cdot \mathrm{CB} \qquad (3)$$

Establishing a metric relation where the follow-up AB has a length of 1u.m. (1 unit of measure) (ie AB = 1), we obtain that ⟦(AC)⟧ ^ 2 = CB, so since the total length of segment AB is 1u.m., then the sum of AC segments with CB must be equal to that unit. Therefore, one can write:

$$\mathrm{AC} + \mathrm{CB} = 1.(4)$$

Or, by replacing the term CB with (AC) 2, we have:

$$(\mathrm{AC})^{2} + \mathrm{AC} = 1.(5)$$

Since this last expression corresponds to a quadratic equation, it is observed that by calculating its roots, we will have for the value of the segment AC, given by:

$$AC = \frac{1 \pm \sqrt{5}}{2} \qquad (6)$$

That is,

$$AC' = \frac{1+\sqrt{5}}{2} = \Phi = 1{,}6180339 \ldots = \frac{AC}{CB} \qquad (7)$$

The number Φ = 1.6180339887498948482045868343656338 ... = AC / CB represents the irrational number, known as "gold number" in honor of Phidias (490-430a.C.), A phenomenal Greek sculptor who has always used the golden ratio in his construction.

III. GOLDEN RATIO FOR TWO CONCENTRIC CIRCLES C_1 e C_2

In this section we will use the golden ratio considering that point A (0,0) represents the center of two circles $C_1$and $C_2$with B (x_0, y_0) ∈C_1 and C (x, y) ∈C_2, so that the points considered to be collinear. Let us consider that the segments ¯AB, ¯BC and ¯AC obeys a golden ratio. In this case, we must,

$$|\overline{AB}|^2 = |\overline{AC}|.|\overline{CB}| \qquad (8)$$

Since B (x_0, y_0) belongs to the circumference C_1, we must

$$r_1^2 = x_0^2 + y_0^2$$

https://dx.doi.org/10.22161/ijaers.611.45 ISSN: 2349-6495(P) | 2456-1908(O)

From where we obtain that,

$$|\overline{AB}| = \sqrt{x_0^2 + y_0^2} = r_1 \quad (9)$$

Similarly, since point C (x, y) belongs to the circumference C_2, it follows that

$$r_2^2 = x^2 + y^2 \rightarrow$$

$$|\overline{AC}| = \sqrt{x^2 + y^2} = r_2 \quad (10)$$

We also have that the difference between the coordinates between the points C (x, y) and $B(x_0, y_0)$ can be represented by the following vector,

$$\overline{CB} = (x, y) - (x_0, y_0) = (x - x_0)i + (y - y_0)j \quad (11)$$

Making the module, we get,

$$\overline{CB} = \sqrt{x^2 - 2xx_0 + x_0^2 + y^2 - 2yy_0 + y_0^2} = \sqrt{r_2^2 + r_1^2 - 2(xx_0 + yy_0)} \quad (12)$$

Where do we consider

$$r_2^2 = x^2 + y^2 \text{ e } r_1^2 = x_0^2 + y_0^2 \quad (13)$$

Since the points A, B and C are collinear, it is worth the relation,

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0 \leftrightarrow y = x.\frac{y_0}{x_0} \quad (14)$$

Taking (13) into (12), we obtain,

$$\overline{CB} = \sqrt{r_2^2 + r_1^2 - 2.\left(xx_0 + \frac{y_0^2}{x_0}x\right)} = \sqrt{r_2^2 + r_1^2 - \frac{2x(x_0^2 + y_0^2)}{x_0}} \quad (15)$$

$$\overline{CB} = \sqrt{r_2^2 + r_1^2 - \frac{2r_1^2}{x_0}x} \quad (16)$$

Taking (12), (10) and (9) into (8), we obtain:

$$r_1^2 = r_2.\sqrt{r_2^2 + r_1^2 - \frac{2r_1^2x}{x_0}} \quad (17)$$

Rising to the square (17), it comes that,

$$r_1^4 = r_2^2\left(r_2^2 + r_1^2 - \frac{2r_1^2x}{x_0}\right) = r_2^4 + r_2^2r_1^2 - \frac{2r_1^2r_2^2}{x_0}.x \rightarrow$$

$$2r_1^2r_2^2.\frac{x}{x_0} = r_2^4 - r_1^4 + r_2^2r_1^2$$

$$\rightarrow x = x_0\left(\frac{r_2^4 - r_1^4 + r_2^2r_1^2}{2r_2^2r_1^2}\right) \quad (18)$$

Denoting,

$$K = \frac{r_2^4 - r_1^4 + r_1^2r_2^2}{2r_1^2r_2^2} \quad (19)$$

Taking (19) in (18), we obtain the point x,

$$x = Kx_0 \quad (20)$$

Using this expression in (14), we obtain,

$$y = x.\frac{y_0}{x_0} \rightarrow y = K.x_0.\frac{y_0}{x_0} \rightarrow y = K.y_0 \quad (21)$$

In this case, point C is represented by:

$$C = (x, y) = (Kx_0, Ky_0) = K.(x_0, y_0) \rightarrow$$

$$C(x, y) = K.B(x_0, y_0) \quad (22)$$

What shows that there is a linear combination between the points of C_1 and C_2

3.1 Values assigned to K.

Let points A, B and C be

$$A(0,0)B(x_0, y_0)C(Kx_0, Ky_0)$$

As it was verified the segments ¯AB, ¯BC and ¯AC obey a golden ratio. Thus, we will determine the values of K.

Soon,

$$\overline{AB}^2 = |\overline{AC}||\overline{BC}| \rightarrow x_0^2 + y_0^2$$

$$= \sqrt{K^2x_0^2 + K^2x_0^2}.\sqrt{(K - 1)^2x_0^2 + (K - 1)^2 + y_0^2} \rightarrow$$

$$x_0^2 + y_0^2 = |K|\sqrt{x_0^2 + y_0^2}.|K - 1|\sqrt{x_0^2 + y_0^2} \rightarrow$$

$$x_0^2 + y_0^2 = |K||K - 1||x_0^2 + y_0^2| \rightarrow$$

$$|K||K - 1| = \frac{x_0^2 + y_0^2}{x_0^2 + y_0^2} = 1 \leftrightarrow |K||K - 1| = 1 \rightarrow$$

$$K(K - 1) - 1 = 0 \rightarrow K^2 - K - 1 = 0 \quad (23)$$

The expression (23) represents the relation that leads to the condition predicted by the golden ratio. Therefore,

$$K' = \frac{1 + \sqrt{5}}{2} = a \quad (24)$$

$$K'' = \frac{1 - \sqrt{5}}{2} = b \quad (25)$$

Since the golden ratio as the division of two distances, thus positive, does not make sense to discuss the second solution given by (25). In this case, let us consider K = (1 + √5) / 2 = a as the solution.Thus taking the relation given by (24) in (19), it follows that,

$$a = \frac{r_2^4 - r_1^4 + r_2^2r_1^2}{2r_2^2r_1^2} \quad (26)$$

The expression given by (26) may be useful for calculating concentric circumferential radii to C_1. This is what we will discuss in the next section. Another fact to be considered in this question is that the equation given by (26) has an important application when considering that the planets have circumferential orbits. In this case, it is possible to obtain the values of the planetary distances. Another fact to consider is that the equation given by (26) assumes a proportionality between the given radii. This proportionality factor is the root of the golden ratio. Thus, it becomes possible to obtain distances from the planets in a much easier way.

IV. THE ROOTS OF GOLDENPROPORTION AND PLANETARYDISTANCES.

Taking the relation given by (26). Thisis,

$$a = \frac{r_2^4 - r_1^4 + r_2^2r_1^2}{2r_2^2r_1^2}$$

Explaining $r_2$as a function of r_1, we obtain that, $\left(\frac{r_2}{r_1}\right)^2 - \left(\frac{r_1}{r_2}\right)^2 = 2a - 1 \quad (27)$

https://dx.doi.org/10.22161/ijaers.611.45 ISSN: 2349-6495(P) | 2456-1908(O)

Denoted,

$$\delta = \frac{r_2}{r_1} \quad (28)$$

Taking (27) into (26), we obtain that,

$$\delta^2 - \delta^{-2} = 2a - 1 \quad (29)$$

Or,

$$\delta^4 - (2a-1)\delta^2 - 1 = 0 \quad (30)$$

Substituting (30) the first root of the golden ratio, a = 1.618034, we obtain that,

$$\delta^4 - 2{,}2361\delta^2 - 1 = 0. \quad (31)$$

Extracting the root of this biquadrated equation, where only real solutions are considered, we obtain that,

$$\delta = \frac{r_2}{r_1} = 1{,}618034. \quad (32)$$

Or,

$$r_2 = 1{,}618034 r_1. \quad (33)$$

Substituting (26) for the second root of the golden ratio, a = -0.618034, we obtain that,

$$\delta^4 + 2{,}2361\delta^2 - 1 = 0. \quad (34)$$

Soon

$$r_2 = 0{,}618034 r_1. \quad (35)$$

The equation given by (33) will be useful to obtain the planetary distances. In this case, let us take as reference the average mercury distance to the sun.

4.1 Applications to the Uranus and Neptune planets

4.1.1 Calculation of the minimum distance of Neptune.
Using the equation given by (33) and $r_1 = 18{,}2766AU$(**Table 3**)
as the least distance from Neptune to the sun, we must,

$$r_2 = 1{,}618034 r_1 = 1{,}618034.18{,}2766 = 29{,}572161$$

Soon,

$$r_2 = 29{,}572161$$

Looking still at table 3, we have to$r_2 = 29{,}5711AU$
The relative error for this value will be,

$$Error = \frac{Exactly - Valor\ Approx.}{Exactly}$$

So,

$$Error = \frac{29{,}5711 - 29{,}5722}{29{,}5711} = 0{,}00003719$$

What is equivalent to a 0,004%

4.1.2 Calculation of the maximum distance of Neptune
Using the equation given by (33) and being the value of the orbit of Neptune and considering that it supposes r_1 is the maximum distance of the planet Uranus with$r_1 = 20{,}0874AU$ (**Table 3**).

$$r_2 = 1{,}618034 r_1 = 1{,}618034.20{,}0874 =$$

$$r_2 = 32{,}5020$$

Looking at the table above, we must$r_2 = 30{,}3163$
The relative error for this value will be,

$$Erro = \frac{Exactly - Valor\ Approx.}{Exactly}$$

So,

$$Erro = \frac{30{,}3163 - 32{,}5020}{30{,}3163} =$$

This is equivalent to an error of 3.911%.

Proceeding this way, we can obtain the following Table for the values of the rays given by expression (33) (Table 3).This table expresses the values of the minimum, average and maximum distances according to Kepler's 3rd law and then expresses the values of the distances using the equation 33 from the golden ratio.

Table.1: Comparison of the maximum, average and maximum distances in relation to the golden ratio.

Planets	Minimumdistance (UA)	AverageDistance (UA)	Maximum Distance(UA)	Minimumdistanceeq (33)	AverageDistanceeq (33)	Maximum Distanceeq (33)
Mercury	0.3075	0.3871	0.4667	0.3075[1]	0.3871[1]	0.4667[1]
Venus	0.7184	0.7233	0.7282	0.4975	0.6263	0.7551
Earth	0.9833	1.0000	1.0176	1.1624	1.1703	1.1782
Mart	1.3814	1.5237	1.6660	1.5910	1.6180	1.6465
Ceres[1]	2.5468	2.7663	2.9858	2.5351	2.4654	2.6956
Jupiter	4.9510	5.2028	5.4546	4.1208	4.4760	4.8311
Saturn	9.0075	9.5388	10.0701	8.0109	8.4183	8.8257
Uranus	18.2766	19.1820	20.0874	14.5744	15.4341	16.2938
Neptune	29.7993	30.0578	30.3163	29.5722	31.0371	32.5020
Pluto	29.5711	39.4387	49.3063	48.2163	48.6345	49.0528

The figures (Figure 2, Figure 3 and Figure 4) represent the graphsextractedfromórigin and the data of table 3 in ordertoevaluate and compare the planetarydistancesbetween Kepler distances and the results obtainedfromequation 33.

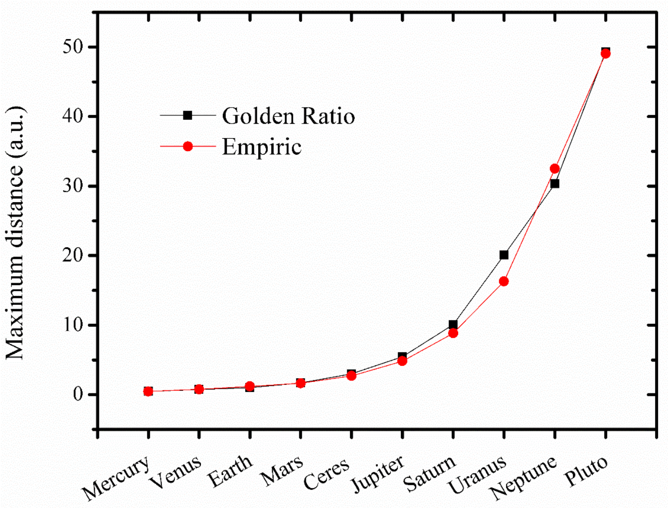

Fig.2: Comparison of Kepler's 1st Kepler's minimum planetary distances with the minimum distances given by Equation-33

Source: Authors' Collection.

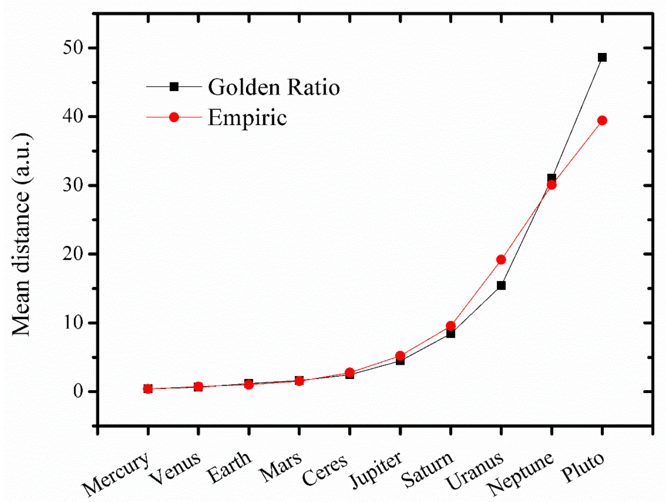

*Fig.3: Comparison of planetary distances Average of Kepler's 1st law with the mean distances given by Equation 33***.**

Source: Authors' Collection.

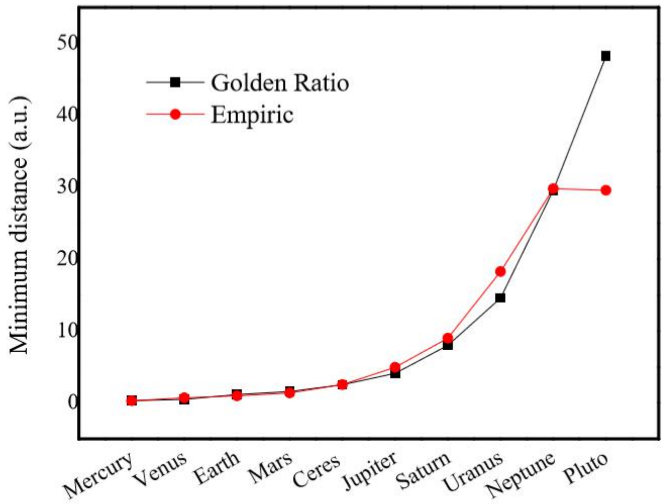

Fig.4: Comparison of Kepler's 1st maximal planetary distances with the maximum distances given by Equation-33.

Source: Authors' Collection.

V. COMPARISONS OF PLANETARY DISTANCES, ACCORDING TO KEPLER, TITUS BODES AND THE GOLDEN RATIO

Table 2: Calculation of planetary distances by Kepler, Titus Bodes and the golden ratio.

Planetas	AverageDistance Kepler (UA)	Average Distance by Titus Bodes (UA)	AverageDistance(Eq.33) (UA)
Mercury	0.39	0.4	0.39
Venus	0.72	0.7	0.63
Earth	1.00	1.00	1.17
Mart	1.52	1.6	1.69
Ceres	2.77	2.8	2.46
Jupiter	5.20	5.20	4.48
Saturn	9.53	10.0	8.42
Uranus	19.18	19.6	15.43
Neptune	30.06	38.8	31.04
Pluto	39.44	77.2	48.63

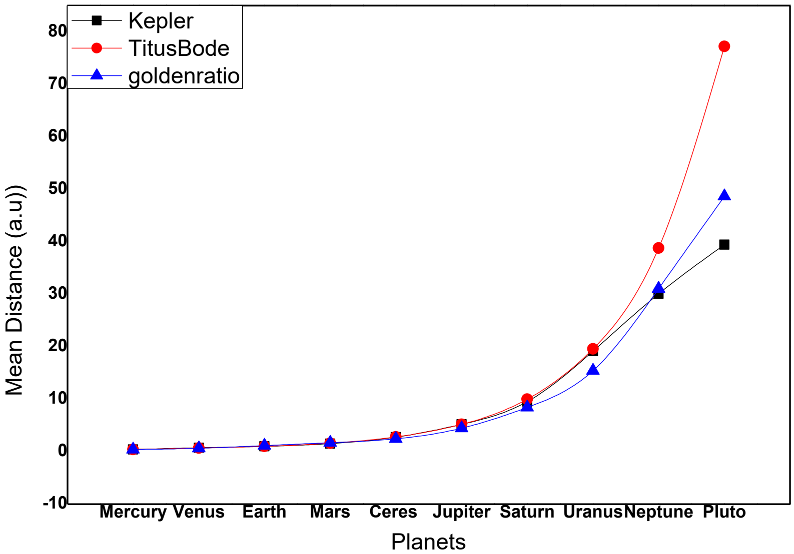

Fig.5: Comparison between planetary distances according to Kepler, Titus Bodes and the golden ratio.
Source: Authors' Collection.

Table 3: Relative error margins of the planetary distances in relation to Titus Bodes, equations (33).

Planets	Average Distance by Titus Bodes (%)	Average Distance per Golden Proportion 1 Eq (33) (%)
Mercury	2.56	0.0
Venus	2.78	12.5
Earth	0.00	17.0
Mart	5.26	11.18
Ceres	1.08	11.19
Jupiter	0.00	13.85
Saturn	4.82	11.65
Uranus	2.08	19.55
Neptune	29.08	3.26
Pluto	95.75	23.30

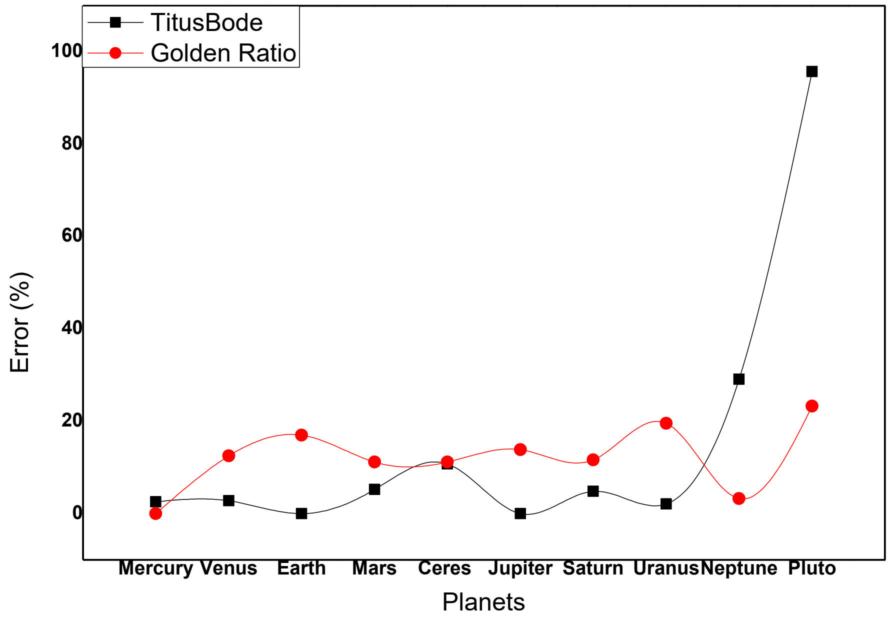

Fig.6: Values of relative errors (%) of the distances of the planets by the law of Titus Bodes and Equations (32) from a golden ratio.

Source: Authors' Collection.

VI. CONCLUSION

As was verified for the planets, uranus and neptune that assume positions with segments that have a certain astronomical relation and that obeys a certain approximation with a golden proportion. The golden ratio has a certain rigor with Kepler's empirical laws and the law of Titus Bodes, presenting in some points a better description than that of Titus Bodes. It can be seen that the equation given by (32) very well describes the astronomical distances when comparing the error margins, as was observed for the uranus and neptune planets. This shows that there is a consistency about the relevance of the golden ratio to these and other planets.

Another fact to consider in this article is that simple idea from geometric theories such as concentric circles was able to show impressive results of astronomical values, as positions of planets near and far in relation to the sun.

For a better description and application of equation (32), graphs were plotted for comparison with Kepler and Titus Bode distances, taking into account comparisons of relative errors. Therefore, with this study, it can be considered that the golden ratio can be used to evaluate the positions of the planets in relation to the sun taking into account the margins of errors to show to what extent the theory has reliability.

Another fact to consider is that simple ideas from geometric theories such as concentric circles, were able to show impressive results of astronomical values, as positions of planets near and far in relation to the Sun.

REFERENCES

[1] Parmanand Singh, The So-called Fibonacci numbers in Ancient and Medieval India, HistóriaMathematica, Vol.12, pp.229–244, 1985.

[2] Swinton J, Ochu E, Consortium TMTS. Novel Fibonacci and non-Fibonacci structure in the sunflower: results of a citizen science experiment. Royal Society Open Science. 2016;3(5). pmid:27293788

[3] Samuel Obara, The Golden Ratio in Art and Architecture, J. Wilson, University of Georgia, Department of mathematical education.

[4] Markowsky, G., O. (1992). Misconceptions about the golden ratio. The College Mathematics Journal, Vol 23, no.1, pp. 2-19

[5] Herz-Fischler, R. (1987). A Mathematical History of the Golden Number.Dover Publications, Inc.

[6] Doczi, Gyorgy (1994). The Power of Limits: Proportional Harmonies in Nature, Art, and Architecture. Boston: Shambhala Publications, Inc.

[7] Md. Akhtaruzzaman* , Amir A. Shafie (2011), Geometrical Substantiation of Phi, the Golden Ratio and the Baroque of Nature, Architecture, Design and Engineering. International Journal of Arts 2011; 1(1): 1-22 DOI: 10.5923/j.arts.20110101.01.

[8] José F. S. Costa, Sebastião M. S. Cordeiro, Abel F. G. Neto, Manuel J. S. Costa, and Antonio M. J. C. Neto (2018), Verification of the Golden Ratio in the Planetary

https://dx.doi.org/10.22161/ijaers.611.45 *ISSN: 2349-6495(P) | 2456-1908(O)*

Orbits of the Solar System, Advanced Science, Engineering and Medicine Vol. 11,pp1–4, 2019
[9] Jaffereys, W.H. and Robins, R.P., Discovering Astronomy, (Toronto: John Wiley, 1981).
[10] Pananides, N.A., Introductory Astronomy, (California: Addison Wesley Publishing Company, 1974
[11] Narlikar, J.V., The Structure of the Universe, (Oxford: Oxford University Press, 1978).
[12] Weinert, Friedel Copernicus, Darwin, & Freud: revolutions in the history and philosophy of science.(Wiley-Blackwell, 2009). ISBN 978-1-4051-8183-9.
[13] Nieto M. M. 1972, TheTitius-Bode Law of Planetary Distances: Its History and Theory (Oxford: Pergamon Press). United Kingdom, England.
[14] Livio, Mario (2002). The Golden Ratio: The Story of Phi, The World's Most Astonishing Number..New York: Broadway Book