International Journal of Advanced Engineering Research and Science (IJAERS)
Peer-Reviewed Journal
ISSN: 2349-6495(P) | 2456-1908(O)
Vol-11, Issue-11; Nov, 2024
Journal Home Page Available: https://ijaers.com/
Article DOI: https://dx.doi.org/10.22161/ijaers.1111.4

Periodic Water Waves: Cnoidal and Solitary Profiles

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Received: 02 Oct 2024,

Receive in revised form: 01 Nov 2024,

Accepted: 08 Nov 2024,

Available online: 12 Nov 2024

***Keywords*— cnoidal-solitary wave profile**

***Abstract*—** *This research formulates the water surface elevation equation for water waves, yielding periodic cnoidal and solitary wave profiles. The equation is derived by integrating the Kinematic Free Surface Boundary Condition with respect to time. The relationships among wave period, wave amplitude, and proportional wavelength facilitate the generation of both cnoidal and solitary wave profiles. In deep water, where wave dynamics are unaffected by the sea bottom, only cnoidal wave profiles are produced. In contrast, solitary profiles emerge through the shoaling-breaking process as waves approach shallower depths.*

I. INTRODUCTION

Wilson (1963) classifies wave profiles into four distinct categories: sinusoidal, Stokes, cnoidal, and solitary profiles. Sinusoidal and Stokes profiles are typically observed in waves with small amplitudes.

The theory of cnoidal waves was first developed by Korteweg and de Vries in 1895, based on observations of wave behavior in canals, thereby confirming their occurrence in natural environments. Cnoidal waves are characterized by a marked asymmetry between the crest and trough, with the wave surface predominantly elevated above the still-water level. The crest presents a steep gradient, while the trough descends below the still-water level more gradually.

Solitary wave profiles, in contrast, are defined by the fact that the entire wave surface remains above the still-water level. This phenomenon was first identified by John Scott Russell in 1844 during laboratory experiments, with the theoretical foundation later provided by Joseph Boussinesq in 1871. Like cnoidal waves, solitary wave profiles have been observed in nature.

Research on cnoidal waves is extensive, with Fenton (1979) contributing significantly to the development of cnoidal wave theory in the context of periodic waves. This research also extends the analysis of periodic cnoidal waves.

Both cnoidal and solitary wave profiles are distinguished by their high crest elevations, with most of the wave profile remaining above the still-water level. In the case of solitary waves, the entire profile stays above this level. Due to their high crest elevations, these wave profiles play a critical role in the design and elevation of coastal structures, as they exert significant forces on such structures. Thus, identifying the appropriate wave profile at a planned construction site is essential.

This research conducts a comprehensive analysis of wave profiles, including assessments in both deep and shallow water, specifically examining wave behavior before and after the breaking point. The water surface elevation equation used for analyzing wave profiles is derived by integrating the Kinematic Free Surface Boundary Condition equation.

II. PRELIMINARY

a. Axis System

In this research, an axis system was used, where x is the horizontal axis and z is the vertical axis.

b. Weighted Taylor Series

The Weighted Taylor series is a truncated form of the Taylor series, limited to the first-order term. In this formulation, coefficients are introduced to the first derivative term, referred to as weighing coefficients.

Weighted Taylor series on function $f(x,t)$,

$$f(x+\delta x,t+\delta t)=f(x,t)+\gamma_{t,2}\delta t\frac{\partial f}{\partial t}+\gamma_x\delta x\frac{\partial f}{\partial x} \quad ......(1)$$

Weighted Taylor series on function $f(x,z,t)$,

$$f(x+\delta x,z+\delta z,t+\delta t)=f(x,t)+\gamma_{t,3}\delta t\frac{\partial f}{\partial t}+\gamma_x\delta x\frac{\partial f}{\partial x}+\gamma_z\delta z\frac{\partial f}{\partial z} \quad ....(2)$$

$\gamma_{t,2}$, $\gamma_{t,3}$, γ_x and γ_z are weighting coefficients. This research employed the equation $\gamma_{t,2}=1.999595$, $\gamma_{t,3}=3.009774$, $\gamma_x=0.997583$ and $\gamma_z=1.022911$. No significant gap is found between γ_x on function $f(x,t)$ and γ_x on function $f(x,z,t)$.

The weighting coefficients and the values of the weighting coefficients were measured based on the formula proposed by Hutahaean (2023).

c. Kinematic free Surface Boundary Condition

Using the weighted Taylor series, the Kinematic Free Surface Boundary Condition is

$$w_\eta=\gamma_{t,2}\frac{\partial\eta}{\partial t}+\gamma_x u_\eta\frac{\partial\eta}{\partial x} \quad ....(3)$$

Or

$$\gamma_{t,2}\frac{\partial\eta}{\partial t}=w_\eta-\gamma_x u_\eta\frac{\partial\eta}{\partial x} \quad ....(4)$$

$\eta(x,t)$ represents the equation for the water surface elevation relative to the still water level, w_η is the surface vertical water particle velocity, and u_η is the surface horizontal water particle velocity.

d. Velocity potential

The velocity potential equation, which is the solution to the Laplace equation (Hutahaean, 2023) under the condition $\sin k_x x=\cos k_x x$, is

$$\phi(x,z,t)=2\,G\cos k_x x\cosh k_z(h+z)\sin\sigma t \quad ....(5)$$

$\phi(x,z,t)$ is velocity potential.

$\sigma=\frac{2\pi}{T}$ is angular frequency and T is wave period.

$k_x=\frac{k}{\sqrt{\gamma_x}}$ is wave number on horizontal axis

$k_z=\frac{k}{\sqrt{\gamma_z}}$ is wave number of the vertical axis

$k_x\approx k_z\approx k$, k is wave number. Although the difference between k_x and k_z is very small, in this research they are still distinguished to maintain calculation accuracy.

G is the wave constant, which, with dimensions of $m.m/sec$, can be referred to as the wave energy transmission rate.

By integrating the kinematic free surface boundary condition, Hutahaean (2024a) obtained the wave amplitude function as:

$$A=\frac{2Gk}{\gamma_{t,2}\sigma}\cosh\theta\pi\left(\frac{\tanh\theta\pi}{\sqrt{\gamma_z}}-\frac{kA}{2}\right) \quad ......(6)$$

θ is the deep water coefficient where $\tanh\theta\pi\approx 1$. In this research, $\theta=3.0$ is used to reduce wave height near the coastline. In previous studies, $\theta=1.94$ was used to obtain $\frac{H_b}{h_b}=0.78$, where H_b is the breaking wave height and h_b is the breaking water depth. With this θ value, a large wave height is observed near and at the coastline, where for a wave with a period of 8 seconds, a wave height of 2.0 meters can occur at a water depth of 1.0 meter.

III. WATER SURFACE ELEVATION EQUATION

The equation for water surface elevation is derived from the integration of the Kinematic Free Surface Boundary Condition with respect to time, utilizing the complete velocity potential.

$$\phi(x,z,t)=G\cosh k_z(h+z)\,(\cos k_x+\sin k_x x)\sin\sigma t \quad ....(7)$$

This equation is substituted into the Kinematic Free Surface Boundary Condition and integrated with respect to time t.

$$\eta(x,t)=\frac{Gk_z}{\gamma_{t,2}\sigma}\sinh\,k_z(h+\eta)\,(\cos k_x x+\sin k_x x)\cos\sigma t+\frac{\gamma_x G\,k_x}{\gamma_{t,2}\sigma}\cosh\,k_z(h+\eta)\frac{\partial\eta}{\partial x}(-\sin k_x x+\cos k_x x)\cos\sigma t \quad ....(8)$$

This equation is highly implicit and nonlinear, where the right-hand side contains $\eta(x,t)$ as a hyperbolic function. The calculations are performed in a stepwise manner as follows.

$$\eta(x,t)=A(\cos k_x x+\sin k_x x)\cos\sigma t$$

$$\frac{\partial\eta}{\partial x}=k_x(-\sin k_x x+\cos k_x x)\cos\sigma t$$

$$\eta(x,t)=\frac{Gk_z}{\gamma_{t,2}\sigma}\sinh\,k_z(h+\eta)\,(\cos k_x x+\sin k_x x)\cos\sigma t+\frac{\gamma_x G\,k_x}{\gamma_{t,2}\sigma}\cosh\,k_z(h+\eta)\frac{\partial\eta}{\partial x}(-\sin k_x x+\cos k_x x)\cos\sigma t \quad .....(9)$$

The wave profile is constructed at a specific value of $\cos\sigma t$ using $\cos\sigma t=1$, over one wavelength, which is defined for $\pi\le k_x x\le 3.0\pi$.

IV. WAVE NUMBER OF DEEP WATER

In this section, the wave number equation for deep water is formulated using the water surface elevation equation, to obtain a wave number that is consistent with the water surface elevation equation.

The maximum water surface elevation is achieved when $\frac{\partial\eta}{\partial x} = 0$. Under this condition, the second term in (9) is zero:

$$\eta_{max} = \frac{Gk_z}{\gamma_{t,2}\sigma}\sinh\, k_z(h+\eta)\,(\cos k_x x + \sin k_x x)\cos\sigma t$$

The maximum water surface elevation relative to its stationary point or still water level is equal to the wave amplitude:

$$A = \frac{Gk_z}{\gamma_{t,2}\sigma}\sinh\, k_z(h+A)\,(\cos k_x x + \sin k_x x)\cos\sigma t$$

The maximum elevation occurs when $\cos\sigma t = 1$.

$$A = \frac{Gk_z}{\gamma_{t,2}\sigma}\sinh\, k_z(h+A)\,(\cos k_x x + \sin k_x x)$$

In (9), the second term is zero if $\cos kx = \sin kx$. Therefore, this condition provides an effect equivalent to $\frac{\partial\eta}{\partial x} = 0$. Under the condition of $\cos kx = \sin kx$,

$$A = \frac{2Gk_z}{\gamma_{t,2}\sigma}\sinh\, k_z(h+A)\cos k_x x$$

If $\cos kx = \sin kx$, the value of $\cos kx = \frac{1}{\sqrt{2}}$

$$A = \frac{\sqrt{2}Gk_z}{\gamma_{t,2}\sigma}\sinh\, k_z(h+A)$$

In the deep water, $k_z(h+A) = k_z h\left(1+\frac{A}{h}\right) \approx k_z h\left(1+\frac{A}{2h}\right) \approx \theta\pi$, where $\tanh\theta\pi \approx 1.0$,

$$A = \frac{\sqrt{2}Gk_z}{\gamma_{t,2}\sigma}\sinh\,\theta\pi$$

Since $k_z = \frac{k}{\sqrt{\gamma_z}}$,

$$A = \frac{\sqrt{2}Gk}{\gamma_{t,2}\sigma\sqrt{\gamma_z}}\sinh\,\theta\pi \qquad \text{.....(10)}$$

The wave amplitude in this equation must be equal to the wave amplitude in (6).

$$\frac{\sqrt{2}Gk}{\gamma_{t,2}\sigma\sqrt{\gamma_z}}\sinh\,\theta\pi = \frac{2Gk}{\gamma_{t,2}\sigma}\cosh\theta\pi\left(\frac{\tanh\theta\pi}{\sqrt{\gamma_z}} - \frac{kA}{2}\right)$$

$$\frac{\tanh\,\theta\pi}{\sqrt{\gamma_z}} = \sqrt{2}\left(\frac{\tanh\theta\pi}{\sqrt{\gamma_z}} - \frac{kA}{2}\right)$$

$$k_0 = \left(2-\sqrt{2}\right)\frac{\tanh\theta\pi}{A_0\sqrt{\gamma_z}} \qquad \text{.......(11)}$$

The equation provided represents the wave number equation applicable in deep water, with the index 0 indicating that the wave amplitude pertains to deep water conditions.

While Equation (11) is derived from Equation (9), it yields a water surface elevation characterized by $\eta_{max} - \eta_{min} < H$. To satisfy the condition $\eta_{max} - \eta_{min} = H$, thus equation (11) must be adjusted by introducing a coefficient given by:

$$k_0 = 1.142x\left(2-\sqrt{2}\right)\frac{\tanh\theta\pi}{A_0\sqrt{\gamma_z}} \qquad \text{......(12)}$$

The coefficient 1.1421 is notably close to 1.0, indicating that Equation (11) aligns well with (9).

V. WAVE PROFILE IN DEEP WATER AT MAXIMUM WAVE HEIGHT

This section focuses on the analysis of wave profiles at maximum wave height during a specific wave period. The maximum wave height in deep water is defined by the Wiegel equation (1949-1964):

$$H_0 = \frac{gT^2}{15.6^2} \qquad \text{.......(13)}$$

Or Hutahaean (2024a),

$$H_0 = \left(\frac{\tanh\theta\pi}{\sqrt{\gamma_z}}\right)^2 \frac{g}{\sigma^2\,\gamma_{t,2}\,\gamma_{t,3}} \qquad \text{.........(14)}$$

Both formulations yield the same maximum wave height.

The calculated values of H_0 and L_0 across various wave periods, along with their wave profile characteristics including critical wave steepness and wave profile criteria based on Wilson (1963) are presented in Table 1.

Table 1: Wave Profile Characteristics in Deep Water

T (sec)	H_0 (m)	L_0 (m)	$\frac{H_0}{L_0}$	$\frac{\eta_{max}}{H_0}$
2	0.161	0.767	0.211	0.864
3	0.363	1.725	0.211	0.864
4	0.646	3.067	0.211	0.864
5	1.009	4.793	0.211	0.864
6	1.453	6.902	0.211	0.864
7	1.978	9.394	0.211	0.864
8	2.583	12.27	0.211	0.864
9	3.269	15.529	0.211	0.864
10	4.036	19.172	0.211	0.864
11	4.884	23.198	0.211	0.864
12	5.812	27.607	0.211	0.864
13	6.822	32.4	0.211	0.864
14	7.911	37.576	0.211	0.864
15	9.082	43.136	0.211	0.864

16	10.333	49.079	0.211	0.864
17	11.665	55.406	0.211	0.864
18	13.078	62.116	0.211	0.864

In Table (1), $\frac{H_0}{L_0}$ represents the wave steepness. Given that H_0 is defined as the maximum wave height, the resulting $\frac{H_0}{L_0}$ reflects the critical wave steepness. Research by Toffoli et al. (2010) established a critical wave steepness threshold of 0.170, recommending a value of 0.200. The critical wave steepness derived from this research is $\frac{H_0}{L_0} = 0.211$, indicating that the wavelength obtained aligns closely with the findings of Toffoli et al.

$\frac{\eta_{max}}{H_0}$ represents the wave profile criteria as defined by Wilson (1963). This criterion is detailed in Table 2 and illustrated in Fig 1. The value $\frac{\eta_{max}}{H_0}$ is 0.864, which, according to the Wilson criterion in Table 2, classifies the wave profile as a cnoidal profile.

Table 2: Water wave profile criteria (Wilson (1963)))

Wave type	$\frac{\eta_{max}}{H}$
Airy/sinusoidal waves	< 0.505
Stoke's waves	$0.505 - 0.635$
Cnoidal waves	$0.635 - 1$
Solitary waves	$= 1$

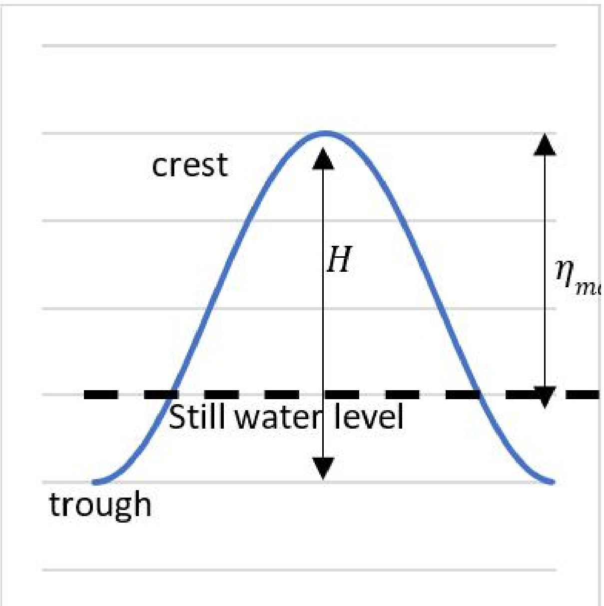

Fig.1: Wave profile for Wilson (1963) criteria.

In Figure 2, the wave profile is illustrated for a wave period of $T = 8.0$ sec with a wave height of $H_0 = 2.583\ m$. The resulting wave profile is identified as a cnoidal profile.

Waves with wave period $T = 2.0\ sec.$ and wave height $H_0 = 0.161\ m.$ are also cnoidal (Fig(3)).

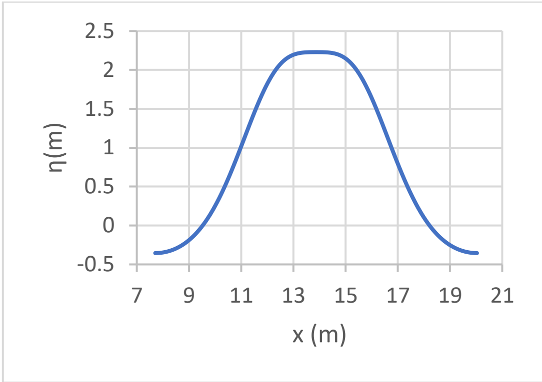

Fig.2: Wave profile, wave period $T = 8.0$, $H_0 = 2.583\ m$, $\frac{\eta_{max}}{H_0} = 0.864\ m.$

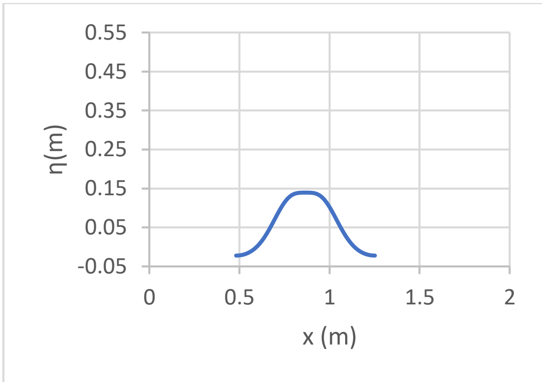

Fig.3: Wave profile, wave period $T = 2.0$, $H_0 = 0.161\ m$, $\frac{\eta_{max}}{H_0} = 0.864.$

The findings of this section indicate that the wave profile generated by the system of equations specifically, the maximum wave height equation, the wave number equation, and the water surface elevation equation in deep water yields a cnoidal wave profile.

VI. SHOALING-BREAKING MODEL

The shoaling-breaking model employed in this research is based on the framework developed by Hutahaean (2023). As waves propagate from point x with water depth h_x towards $x + \delta x$, at small δx, with water depth $h_{x+\delta x}$ show changes in the parameters as follows.

$$\frac{\partial k}{\partial x} = -\frac{4k}{(4h+3A)}\frac{dh}{dx} \qquad \text{……(15)}$$

$$k_{x+\delta x} = k_x + \delta x \frac{\partial k}{\partial x}$$

$$\frac{\partial A}{\partial x} = \frac{G}{\sigma \gamma_{t,2}} \frac{\partial k}{\partial x} \left(\frac{1}{\sqrt{\gamma_z}} - \frac{kA}{2} \right) \cosh(\theta\pi) \qquad \text{…….(16)}$$

$$A_{x+\delta x} = A_x + \delta x \frac{\partial A}{\partial x}$$

$$G_{x+\delta x} = e^{\ln G_x - \frac{1}{2}(\ln k_{x+\delta x} - \ln k_x)} \qquad \text{……..(17)}$$

a. Results of Shoaling-Breaking Analysis with Wave Profile

In this section, a shoaling-breaking analysis is conducted on waves characterized by a wave period of $T = 8.0\, sec.$, wave height $H_0 = 2.583m.$, deep water depth $h_0 = 17.759$ m. The findings from this analysis are illustrated in Figure 4, where the breaking wave height is $H_b = 3.298$ m at a breaker depth $h_b = 6.921\ m$, $\frac{H_b}{h_b} = 0.477$. In this research, the traditional criterion of $\frac{H_b}{h_b} = 0.78$, has been set aside, as adherence to this standard resulted in excessively large wave heights near the coastline. If the criterion of $\frac{H_b}{h_b} = 0.78$ d were to be utilized, it would necessitate a deep water coefficient $\theta = 1.94$. However, to ensure that the wave height near the coastline remains manageable, a deep water coefficient of $\theta = 3.0$ was employed in this analysis.

It is important to note that the breaking wave height observed in this research differs from findings in previous research, such as Hutahaean (2024b). This discrepancy can be attributed to the longer wavelength in deep water present in this research, which consequently leads to an increase in wave energy.

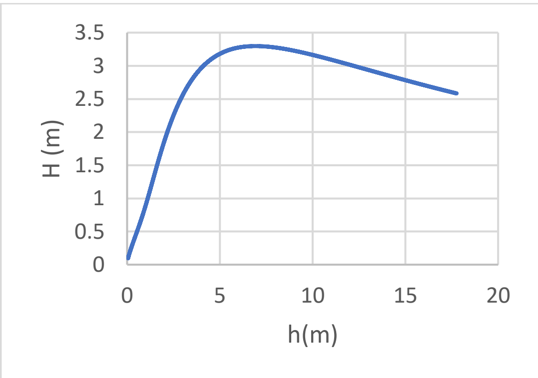

Fig.4: The results of shoaling-breaking analysis

Subsequently, a wave profile analysis was carried out in shallow water at a water depth. h, 12.0 m, 6.921 m (breaking point), 3.0 m ,1.0 m and 0.50 m .

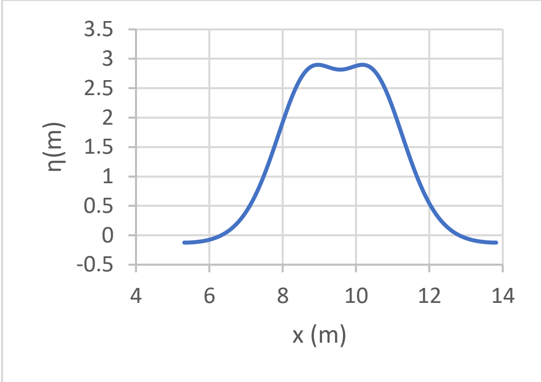

Fig.5: Wave profile at $h = 12.0\ m$, $\frac{\eta_{max}}{H} = 0.959$, $H = 3.016\ m$

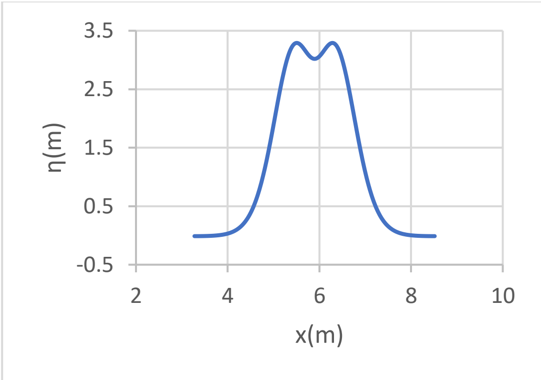

Fig.6: Wave profile at $h = 6.921\ m$ (breaker depth), $\frac{\eta_{max}}{H} = 1.0$, $H = 3.298\ m$

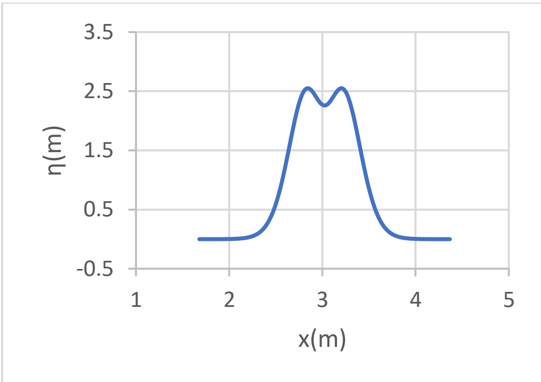

Fig.7: Wave profile at $h = 3.0\ m$, $\frac{\eta_{max}}{H} = 1.0$, $H = 2.560\ m$

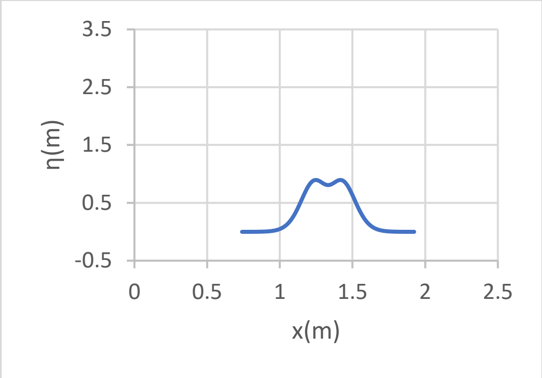

Fig.8: Wave profile at $h = 1.0\ m, \frac{\eta_{max}}{H} = 1.00, H = 0.895\ m$

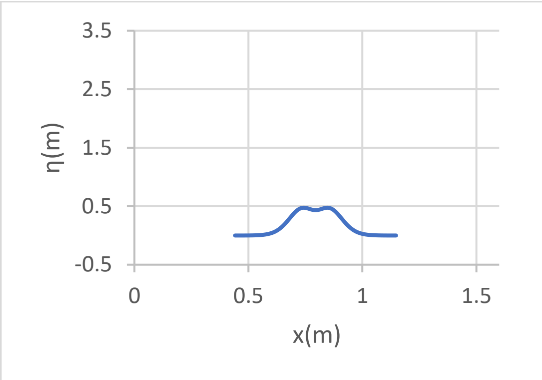

Fig.9: Wave profile at $h = 0.5\ m, \frac{\eta_{max}}{H} = 1.00, H = 0.475\ m$

Table 3: Wave Profile Summary

h (m)	H (m)	$\frac{\eta_{max}}{H}$	Profile
17.87	2.583	0.864	Cnoidal
12.0	3.016	0.959	Cnoidal
6.921	3.298	1.0	Solitary
3.0	2.560	1.0	Solitary
1.0	0.895	1.0	Solitary
0.5	0.475	1.0	Solitary

In deep water, as illustrated in Figure 2, the wave profile is characterized as cnoidal, with a ratio of $\frac{\eta_{max}}{H_0} = 0.864$. As the wave progresses towards the coastline, this cnoidal profile undergoes significant evolution, primarily through an increase in the value of $\frac{\eta_{max}}{H}$ as presented in Table 2. This transformation ultimately leads to the formation of a solitary wave profile. At the breaking point, where $\frac{\eta_{max}}{H} = 0.996$ the wave can be classified distinctly as a solitary profile.

In a time series model, Hutahaean (2024b) corroborated these findings, demonstrating that the cnoidal profile observed in deep water evolves into a solitary profile as it transitions into shallower waters.

VII. CONCLUSION

The first conclusion drawn from this research is that the system of equations, which includes the dispersion equation, wave height equation, and surface elevation equation, generates both cnoidal and solitary wave profiles. These profiles are characteristic of short waves commonly observed in nature.

As cnoidal waves propagate from deep to shallow water, they undergo a profile evolution, transitioning from a cnoidal to a solitary wave form. This evolution is marked by an increase in the parameters governing the wave profile.

In shallow water, wave profiles are predominantly cnoidal and/or solitary. For structural design in shallow water, it is recommended to use the solitary wave profile for both elevation planning and wave force calculations. In contrast, for deep-water conditions, the cnoidal profile can be applied when the Wilson criterion exceeds 0.8. However, for enhanced safety, it is advisable to use the solitary wave profile, regardless of the water depth, to account for potential extreme wave forces.

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