International Journal of Advanced Engineering Research and Science (IJAERS) *[Vol-3, Issue-10, Oct- 2016]*
https://dx.doi.org/10.22161/ijaers/3.10.5 *ISSN: 2349-6495(P) | 2456-1908(O)*

Chiral Wave Modes in Elliptical Region

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Abstract—*The linear theory of chiral surface waves in cylindrical containers with an elliptical cross-section is studied in detail. General solutions for phase and the free amplitude are given in terms of Mathieu functions. Our numerical results show the dependence of the numberwaves and natural frequencies on the chiral media with eccentricity* ε. *The well known case of a circular media for* $\varepsilon = 0$ *is retrieved and remarkable crossings of the mode frequencies for certain values of* ε *are found. The frequency shift is evaluated numerically.*

Keywords— elliptical region, Chiral wave, membrane oscillation.

I. INTRODUCTION

The problems of membrane oscillation have been the subject of many papers [1-4]. The theoretical and numerical investigations presented in these papers concern rectangular, circular and arbitrary shaped media. In these cases, non-homogeneous problems can be solved by using Green's function method. The problem of oscillation of a non-uniform or non-regular shaped system is solved by using approximate (numerical) methods [5]. The method of fundamental solutions can be used as an example of such approximate methods [4]. For chiral media of regular shapes (rectangle, circle), an exact solution to electromagnetic problems can be derived.

The application of Green's function method to the oscillation problem of a chiral media, which occupies a finite region in the plane, requires the knowledge of Green's function for the Helmholtz equation. The method of fundamental solutions applied to electromagnetic problems of arbitrary shaped also requires knowledge of the fundamental solution of the Helmholtz equation in the plane. Green's functions of the Helmholtz equation in regular regions are well known. These functions for problems in rectangular and circular regions with various boundary conditions are given in [6-8]. In this paper, the derivation of Mathieu's function for the chiral Helmholtz equation in an elliptical region is presented. In order to separate the variables, elliptical coordinates are introduced. The solution is obtained in the form of a series of eigen functions of the associated boundary problem.

II. CHIRAL ELECTROMAGNETIC FORMULATION

The Maxwell equations for the macroscopic free electromagnetic fields, (without charge and current) are well known. We often write Maxwell's equations in terms of electric and magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$,

$$\nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}\,,\ \nabla\cdot\mathbf{B} = 0 \tag{1a}$$

$$\nabla\times\mathbf{H} = \frac{\partial}{\partial t}\mathbf{D}+\mathbf{J}\,,\ \nabla\cdot\mathbf{D} = \rho \tag{1b}$$

These equations, however, are not complete. Six more equations, the constitutive relations, have to be added relating the electric field $\mathbf{E}$, the magnetic induction $\mathbf{B}$, the displacement field $\mathbf{D}$ and the magnetic field $\mathbf{H}$ to each other. These constitutive relations are completely independent of the Maxwell equations. The Maxwell equations involve only the fields and their sources. The constitutive relations, however, are concerned with the equations of motion of the constituents of the medium in an electromagnetic field [9-14].

We often write Maxwell's equations in terms of electric and magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$,
defined by with the non locality definitions of Born-Fedorov, [9-11]:

$$\mathbf{B} = \mu_c\left[\mathbf{H} + T^c(\nabla\times\mathbf{H})\right] \tag{2}$$

$$\mathbf{D} = \epsilon_c\left[\mathbf{E} + T^c(\nabla\times\mathbf{E})\right] \tag{3}$$

where T^c is the chiral factor. The wave propagation is

$$(1-\omega^2\mu_c\varepsilon_c T^{c2})\nabla\times(\nabla\times\mathbf{E})\text{-}2T^c\mu_c\varepsilon_c\omega^2\nabla\times\mathbf{E}\text{-}\omega^2(\mu_c\varepsilon_c)\mathbf{E}=\mathbf{0} \tag{4}$$

https://dx.doi.org/10.22161/ijaers/3.10.5 *ISSN: 2349-6495(P) | 2456-1908(O)*

The particular case with $1-\omega^2\mu_c\varepsilon_c T^{c2}=0$ corresponds to $\mathbf{E}\parallel\mathbf{B}$ with $\mathbf{E}=\mathbf{i}\,\mathbf{H}$ so we have

$$2T^c\nabla\times\mathbf{E}-\mathbf{E}=0 \tag{5}$$

which satisfy the chiral homogeneous Helmholtz equation

$$(\nabla^2\mathbf{E})+\left(\frac{1}{2T^c}\right)^2\mathbf{E}=0 \tag{5'}$$

For each component of $\mathbf{E}$ or $\mathbf{H}$, is valid the equation

$$(\nabla^2\Phi)+\left(\frac{1}{2T^c}\right)^2\Phi=0 \tag{6}$$

This result corresponds to self-dual solutions to the Maxwell equations. These solution are known as “instantons,” have gained recognition among experts in gauge field theory and mathematical physics. The condition that a field configuration is self-dual is not invariant under the parity transformation $\mathbf{r}\to\mathbf{-r}$ because of the opposite parity properties of the electric and magnetic field; the mirror-image configuration is anti-self-dual. As will become clear, the physically relevant configurations are represented by a sum of self-dual and anti-self-dual solutions, which is invariant under the parity transformation. In this paper we discuss this type of configuration in elliptical geometry in two dimensions.

III. PROBLEM FORMULATION IN ELLIPTICAL COORDINATES

To solve the problem, we introduce elliptical coordinates in. the interior of the domain D (χ,θ), which are coupled with Cartesian coordinates by the following relationships (Fig. 1):

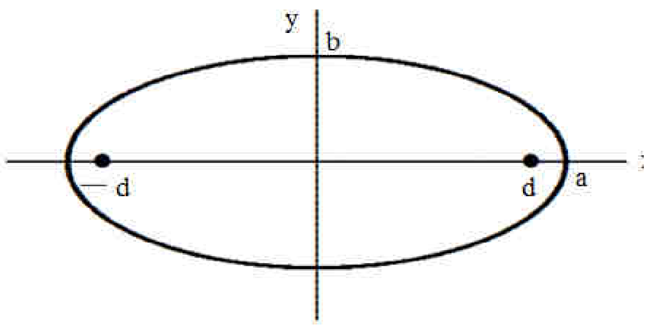

Fig.1: Introduce elliptical coordinates (χ,θ) from Cartesian coordinates.

$$x=d\cosh\chi\cos\theta$$

$$y=d\sinh\chi\sin\theta \tag{7}$$

where $\chi\geq 0$, $0\leq\theta\leq 2\pi$ □. The equation of ellipse (7) in the elliptical coordinates is: $\chi=\chi_0$, where $\chi_0=\mathrm{artgh}\frac{b}{a}$. The Laplace operator in elliptical coordinates has the form

$$\nabla^2 U=\frac{1}{d^2(\cosh^2\chi-\cos^2\theta)}\left(\frac{\partial^2 U}{\partial\chi^2}+\frac{\partial^2 U}{\partial\theta^2}\right) \tag{8}$$

Differential equation (1) and boundary condition (2) in the elliptical coordinates
are as follows

$$\frac{1}{d^2(\cosh^2\chi-\cos^2\theta)}\left(\frac{\partial^2}{\partial\chi^2}+\frac{\partial^2}{\partial\theta^2}\right)U=-\left(\frac{1}{2T_c}\right)^2 U \tag{9}$$

$$U(\chi_0,\theta,T_c)=0,\ \text{for } 0\leq\theta\leq 2\pi \tag{10}$$

The solution of boundary problem (9), (10) can be found in the series form

$$U(\chi,\theta,T)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\Phi_{mn}(\chi,\theta)T_{mn}(T_c) \tag{11}$$

where functions $\Phi_{mn}(\chi,\theta)$ satisfy the homogeneous Helmholtz equation

$$\left(\frac{\partial^2}{\partial\chi^2}+\frac{\partial^2}{\partial\theta^2}\right)\Phi_{mn}(\chi,\theta)+\left(\frac{1}{2T^c_{m,m}}\right)^2 d^2(\cosh^2\chi-\cos^2\theta)\Phi_{mn}(\chi,\theta)=0 \tag{12}$$

In our case we are looking for solutions of Eq. (12) in the interior of the domain D with boundary condition

$$\Phi_{mn}(\chi_0,\theta,T^c)=0,\ \text{ for } 0\leq\theta\leq 2\pi \text{ and } 0\leq\chi\leq\chi_0 \tag{13}$$

In order to derive eigenfunctions $\Phi_{mn}(\chi,\theta)$, the method of separation of variables will be used. We assume that

$$\Phi_{mn}(\chi,\theta)=NR_m(\chi,q_{mn})\psi(\theta,q_{mn}) \tag{14}$$

where N is a normalization constant. After the substitution of (11) into equation (9) and separation of the variables, two equations are obtained:

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$$\frac{d^2R_m}{d\chi^2} - (a - 2q_{mn}\cosh 2\chi)R_m = 0 \tag{15}$$

$$\frac{d^2\psi_m}{d\theta^2} - (a - 2q_{mn}\cos 2\theta)\psi_m = 0 \tag{16}$$

where *a* is the separation constant and $(1/2T^c_{m,n})^2 = 2q_{mn}/d^2$. From physical considerations the function $\psi(\theta)$ must be periodic, i.e. $\psi(\theta) = \psi(\theta + 2\pi)$.Taking into account boundary condition (13) and equation (14), we found that q_{mn} are the roots of the equation

$$R_m(\chi_0, q_{mn}) = 0 \tag{17}$$

Moreover, we assume that functions $\psi_m(\theta, q_{mn})$ are periodic with period □□or 2□. This property holds for particular values of separation constant *a*, which depends on the values of q_{mn} .

Equations (15) and (16) are well known as the radial Mathieu equation and the angular Mathieu equation respectively. The pairs of the independent solutions of these equations are radial and angular Mathieu functions [2]:

$$R_m(\chi, q_{mn}) = \begin{cases} Ce_m(\chi, q_{mn}) \\ Se_{m+1}(\chi, q_{mn}) \end{cases} \tag{18a}$$

where Ce_m and Se_{m+1}Sem are the modified Mathieu functions of first kind or radial solutions.

$$\psi_m(\theta, q_{mn}) = \begin{cases} ce_m(\theta, q_{mn}) \\ se_{m+1}(\theta, q_{mn}) \end{cases} \tag{18b}$$

$m = 0, 1, 2.....$

Using (11) and introducing functions

$$me_{2m}(\theta, q) = ce_m(\theta, q),\quad me_{2m+1}(\theta, q) = se_{m+1}(\theta, q),\quad m = 0, 1, 2.. \tag{19}$$

$$Me_{2m}(\chi, q) = Ce_m(\chi, q),\quad Me_{2m+1}(\chi, q) = Se_{m+1}(\chi, q) \tag{20}$$

function $\Phi_{mn}(\chi, \theta)$ can be written in the form

$$\Phi_{mn}(\chi, \theta) = Me_m(\chi, q_{mn}) me_m(\theta, q_{mn}) \tag{21}$$

The angular Mathieu functions create the set of the orthogonal system in interval [0,2□], i.e. the following orthogonality condition holds:

$$\frac{1}{\pi}\int_0^{2\pi} me_m(\theta, q) me_n(\theta, q) d\theta = \delta_{mn} \tag{22}$$

where δ_{mn} □is the Kronecker delta. This leads to the statement that eigenfunctions $\Phi_{mn}(\chi, \theta)$ given by (11) satisfy the condition

$$\iint_D \Phi_{mn}(\chi, \theta)\Phi_{kl}(\chi, \theta) d\chi d\theta = \pi\delta_{mk}$$

□□□□

Following the spatial symmetry of Eq. (12), the complete set of solutions can be chosen as even or odd eigenfunctions.

IV. EIGENVALUES IN ELLIPTICAL COORDINATES

In elliptic coordinates the condition $\partial R_m(\chi, q_{mn})/\partial\vec{n} = 0$ at (x, y) in 2- D reduces to

$$dR_m(\chi, q_{mn})/d\chi\big|_{\chi=\chi_0} = 0 \tag{24}$$

with $\chi = \operatorname{arctan h}(b/a)$, to work with number wave, let $(1/2T^c_m)d = dk_m = \varepsilon\bar{k}_m$, ε being the eccentricity and $\bar{k}_m = ak_m = a/(2T^c_m)$, then from Eq. (24) we obtain the set of even and odd eigenvalues for $\bar{k}_{m,n}$, where $\bar{k}^{even}_{m,n}$ and $\bar{k}^{odd}_{m,n}$ are solutions of

$$Ce_m(\chi_0, \varepsilon\bar{k}^{even}_{m,n}/2) = 0, \qquad Se_m(\chi_0, \varepsilon\bar{k}^{odd}_{m,n}/2) = 0 \tag{25}$$

with n = 1, 2, ... Recall that the eccentricity e of an ellipse is related to its semi-axes by the expression $\varepsilon = \sqrt{1-(b/a)^2}$ so that one can write $\chi_0 = \operatorname{arc\,tanh}\sqrt{1-\varepsilon^2}$, and therefore,

the characteristic values $\bar{k}_{m,n}$ depends solely on the eccentricityε. Thus, if the ratio value between semi-axes b/a is modified, the pattern of the wave amplitude will be

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different carrying the information of the surface symmetry and thereby of elliptic geometry.

The eigenvalue problem (25) states that for a given m the symmetry (odd or even) remains valid for any eccentricity value. Thus, eigenvalues $\bar{k}_{m,n}$ for even and odd solutions with the same m but different elliptical oscillation number n cannot cross (anticrossing) as a function of the geometry factor ε, i.e. the symmetry is preserved. Moreover, eigensolutions belonging to different Hilbert subspace can cross. These important facts can be clearly seen in Figs. 2 and 3. Fig. 2 shows the variation of the first 10 eigenvalues $\bar{k}_{m,n}$ as a function of ε. Notice that in the limit $\varepsilon \to 0$ the degeneracy of the modes, that corresponds to basins with circular cross sections, is clearly displayed, i.e., $\bar{k}_{m,n} = a/(2T^{c}_{m,n})$ tends to $(a/(2T^{c}_{m,n}))$ with $(\varepsilon = 0)$.

An important conclusion arises from Fig. 2 where it can be seen that $\bar{k}_{m,n}$ increases as the eccentricity increases, i.e. narrowed ellipses or chiral surface waves more confined, lead to larger values of the eigenvalues $\bar{k}_{m,n}$. Moreover and according to the general properties of the Mathieu functions, $\bar{k}^{odd}_{m,n}$ are more sensitive to the confinement than the even eigenvalues $\bar{k}^{even}_{m,n}$, and the inequality $\bar{k}^{odd}_{m,n} > \bar{k}^{even}_{m,n}$ is valid for any eccentricity value

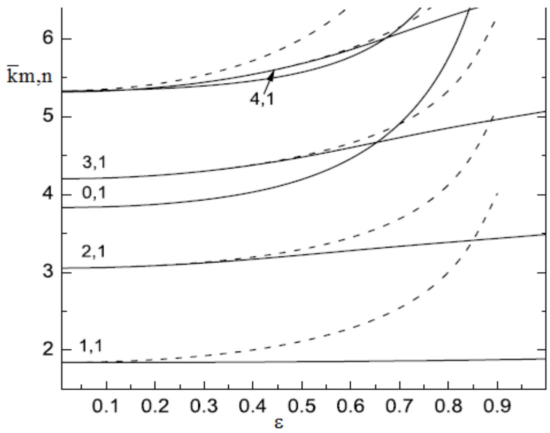

Fig.2: Eigenvalues $\bar{k}_{m,n}$ as a function of the eccentricity e. Dashed (solid) lines represent eigenvalues for the odd (even) modes.

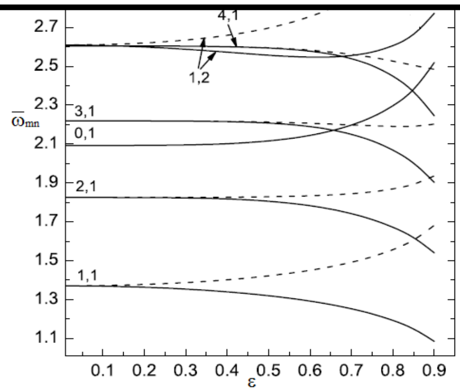

Fig.3: Mode frequencies as a function of the eccentricity ε.

A comparison of the elliptic case with the classical circular basins assuming the same cross-section area shows a strongly changing behavior of the natural frequency $\omega_{m,n}$ as a function of the geometric factor ε.

Fig. 3 is devoted to the evaluation of $\omega_{m,n}$ for the first 10 modes as a function of the geometrical parameter ε. The behavior of $\omega_{m,n}$ versus ε is very diverse. As a general trend, the even modes (m >0, n = 1) show decreasing frequencies as ε increases. Also, it is observed modes reaching to a minimum frequency value for certain ε_{min}, increasing $\omega_{m,n}$ for $\varepsilon > \varepsilon_{min}$, while there are modes exhibiting monotone increasing behavior of $\omega_{m,n}$ as the eccentricity increases (in Fig. 3 the odd modes (1, 1), (1, 2), (2, 2) and the even mode (0, 1)). Notice that for the odd mode (3, 1), $\omega_{m,n}$ is almost independent of ε. A more interesting feature is the accidental degeneracy obtained at certain values of ε. From the figure the crossing points of two different mode frequencies for large values of the eccentricity are clearly observed.

V. CONCLUSIONS

The derivation of the Mathieu function of the chiral wave equation in an elliptical region with the Dirichlet boundary condition has been presented. In order to solve the problem, elliptical coordinates were introduced. The function has the form of a double series of Mathieu functions, which are eigenfunctions of the Helmholtz operator in the considered elliptical region. Although, the solution is obtained for the Dirichlet condition at the boundary ellipse.

The solution to the non-homogenous problem of the chiral oscillation can be presented in an exact form. The linear theory of chiral surface waves in cylindrical containers with an elliptical cross-section was studied. General solutions for phase and the free amplitude are given in terms of Mathieu functions. Our numerical results show the dependence of the natural frequencies on the chiral media and the eccentricity ε of the container cross-section. The well known case of a circular media for $\varepsilon = 0$ is retrieved and remarkable crossings of the mode frequencies for certain values of ε are found. The frequency shift is evaluated numerically.

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