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ON A MODEL OF GENERALIZED PELL NUMBERS

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ABSTRACT. In this study we investigate a model of generalized Pell numbers. Combinatorial representations are provided and some new identities and combinatorial identities are established. Moreover, analytic results are exhibited, where some special cases are discussed. Illustrative examples and applications are given.

Key Words : Model of Generalized Pell numbers, Combinatorial Representation, Combinatorial Identities, Analytic Representations.

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1. INTRODUCTION

Several families of integers defined by recursive relations are studied in the literature. These sequences of numbers are at the origin of many interesting identities, in particular of combinatorial or analytic types. It is possible find in-depth results on these families of integers and their generalizations. Specially among theses families, the Pell numbers $P_n, (n \geq 0)$ is one of the most well-known sequence of integers with important role in various topics of mathematics, and also in exact and applied sciences (see more in [4–6,17]). These integers defined by the initial conditions $P_0 = 0$, $P_1 = 1$ and the classical recurrence relation, $P_{n+1} = 2P_n + P_{n-1} \quad \text{for} \quad n \geq 1$, have been widely discussed from both algebraic, analytical and combinatorial perspective. Furthermore, diverse generalizations of the sequence $\{P_n\}_{n\geq 0}$ of Pell numbers have been considered in various research papers (see for example [4–7,10,11,17]). Such generalizations are defined by recurrence relations of second order. The first proposed generalization is defined by,

$$P_{d,n} = 2^d P_{d,n-1} + P_{d,n-2} \text{ for } n \geq 2,$$

with appropriate initial conditions $P_{d,1} = 1$ and $P_{d,2} = 2^d$ (see [17]), and the second generalization is given by,

$$P_{h,n} = 2P_{h,n-1} + hP_{h,n-2} \text{ for } n \geq 2,$$

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where h is a positive real number, and with appropriate initial conditions $P_{h,0} = \alpha_0$ and $P_{h,1} = \alpha_1$ (see [4–6]). Another type of generalization of Pell numbers is defined by the following linear recursive relation of order $r \geq 2$,

$$P_{n+1} = 2P_n + P_{n-1} + \cdots + P_{n-r+1}$$

for $n \geq r-1$, where the initial conditions $P_0 = \alpha_0$, ..., $P_{r-1} = \alpha_{r-1}$ are chosen adequately (see [12,13,16]). In general, the initial conditions for this kind of Pell numbers are given by $P_0 = ... = P_{r-2} = 0$, $P_{r-1} = 1$.

The present study deals with the model of generalized Pell numbers defined with integer parameters $d \geq 0$, i $(0 \leq i \leq r-2)$, $h \geq 1$ by the following linear difference equation of order $r \geq 2$,

$$P_{d,i,h,n+1} = 2^d P_{d,i,h,n} + P_{d,i,h,n-1} + \cdots + hP_{d,i,h,n-r+1} \tag{1}$$

for $n \geq r$, where the initial conditions $P_{d,i,h,0} = \alpha_0$, ..., $P_{d,i,h,r-1} = \alpha_{r-1}$ are chosen adequately. Expression (1) extends the preceding generalizations. That is, if we set $a_0 = 2^d, a_1 = 1$ and $a_j = 0$ for $2 \leq j \leq r-1$ or $a_0 = 2$, $a_1 = h$ and $a_j = 0$ for $2 \leq j \leq r-1$, or $a_0 = 2$, $a_1 = \cdots = a_{r-1} = 1$, respectively, in $u_{n+1} = \sum_{i=0}^{r-1} a_i u_{n-i-1}$, we get the expressions defining the three preceding generalizations of Pell numbers.

We investigated the model of generalized Pell numbers defined by the difference equation (1) through properties of an associated basic fundamental system. Our approach is also issued from the general setting and properties of r-generalized Fibonacci sequences (see for example, [8,14,18]). These methods allow us to obtain various statements for the generalized model of Pell numbers. More precisely, the combinatorial representation is formulated and some new identities are provided. The analytical representations were studied for some special cases, namely, $i = 0$, $h = 1$ and $i = r-2$, given by the expressions

$$P_{d,0,1,n+1} = 2^d P_{d,0,1,n} + P_{d,0,1,n-1} + \cdots + P_{d,0,1,n-r+1} \tag{2}$$

for $n \geq r$,

$$P_{d,r-2,h,n+1} = 2^d P_{d,r-2,h,n} + hP_{d,r-2,h,n-r+1} \tag{3}$$

for $n \geq r$, with $d \geq 0$ and $h \geq 1$. Furthermore, the analytical representations for the general sitting for two special cases were established, namely, $r = 4, i = 1$ and $r = 5, i = 1$ or $i = 2$, with positive integers d and h.

Note that by taking $d = 0$ we obtain Expression (2) defining the generalized Fibonacci numbers (see more in [15]), and for $d = 1$ Expression (2) is the recursive relation for generalized Pell numbers studied in [16]. For parameters $d = 0$ and $h = 1$, Expression (3) is a recursive relation defining Fibonacci $r-$numbers. Moreover, it is worth noting that for the generalized Pell numbers (3), equipped with the following kind of initial conditions $\alpha_0 = ... = \alpha_i = 0$ or $\alpha_{i+1} = ... = \alpha_{r-1} = 1$ are considered in [12], and labeled the generalized (r, i)-Pell numbers.

The content of this paper is organized as follows. In Section 2 we study the combinatorial representation of the families of sequences of the model of generalized Pell numbers, defined with the aid of the difference equation (1). Moreover, some new identities and combinatorial identities are provided in Section 3. Section 4 concerns a special model of generalized Pell numbers. Also the analytical aspect of the Fibonacci numbers is provided. In addition, properties of the analytical representation of the generalized Pell numbers (3) are exhibited. Section 5 is devoted to analytical expression of some special cases of generalized Pell numbers (1). Finally, discussion and concluding remarks are given.

For reason of simplicity and clarity, in this work we will omit the parameters d and h and the model of generalized Pell numbers $P_{d,i,h,n}$ defined by Expression (1) will be notated in general case by $P_{d,i,h,n} = P_{i,n}$, notated by $P_{d,r-2,h,n} = R_n$ for $i = r-2$ and notated by $P_{d,0,h,n} = P_n$ for $i = 0$.

2. Combinatorial representation of the model of generalized Pell numbers

The preceding generalizations represent special cases of the sequences $\{u_n\}_{n\geq 0}$ defined by,

$$u_{n+1} = \sum_{i=0}^{r-1} a_i u_{n-i-1} \qquad \text{for} \qquad n \geq r, \tag{4}$$

known in the literature as linear difference equation of constant coefficients $a_i \in \mathbb{R}$ or $\mathbb{C}$ $(0 \leq i \leq r-1)$. When the initial data $u_0 = \alpha_0$, $u_1 = \alpha_1$,..., $u_{r-1} = \alpha_{r-1}$ are specified, sequences defined by the recursive relation (4) are known in the literature as *r-generalized Fibonacci sequences*. It is well known that the combinatoric formula of sequences (4) have been largely studied in the literature (see, for example, [14, 18] and references therein). Indeed it was shown in [14] that,

$$u_n = \rho(n,r)A_0 + \rho(n-1,r)A_1 + \cdots + \rho(n-r+1,r)A_{r-1}, \text{ for every } n \geq r, \tag{5}$$

such that $A_m = a_{r-1}u_m + \cdots + a_m u_{r-1}$ and

$$\rho(n,r) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-r} \frac{(k_0+\cdots+k_{r-1})!}{k_0!k_1!\ldots k_{r-1}!} a_0^{k_0} a_1^{k_1} \ldots a_{r-1}^{k_{r-1}}, \text{ for every } n \geq r, \tag{6}$$

where $\rho(j,r) = 0$ for $0 \leq j \leq r-1$ and $\rho(r,r) = 1$. When in Expression (4) the coefficients are given by $a_0 = 2^d$, $a_1 = \cdots = a_i = 0$; $a_{i+1} = \cdots = a_{r-2} = 1$ and $a_{r-1} = h$ and the initial conditions $\alpha_0, \cdots, \alpha_{r-2}, \alpha_{r-1}$, we get Expression (1) defining the model of generalized Pell numbers. Therefore, the construction did in [14] and Expression (6) implies the following result on the combinatorial aspect of the model of generalized Pell numbers (1).

Theorem 2.1. *Let the generalized Pell numbers $P_{i,n}$ defined by Expression* (1) *with arbitrary initial conditions $\alpha_0, \ldots, \alpha_{r-1}$. Then, for $i = 0$ we have*

$$P_n = \rho_0(n,r)A_0 + \rho_0(n-1,r)A_1 + \cdots + \rho_0(n-r+1,r)A_{r-1},$$

for every $n \geq r$, such that $A_0 = 2^d\alpha_{r-1} + \sum_{k=1}^{r-2}\alpha_k + h\alpha_0$, $A_1 = h\alpha_1 + \sum_{k=2}^{r-1}\alpha_k, \ldots, A_{r-1} = h\alpha_{r-1}$, and

$$\rho_0(n,r) = \sum_{k_0+\cdots+rk_{r-1}=n-r} \frac{(k_0+\cdots+k_{r-1})!}{k_0!\ldots k_{r-1}!} 2^{dk_0} h^{k_{r-1}},$$

for every $n \geq r$, where $\rho_0(j,r) = 0$ for $j \leq r-1$ and $\rho_0(r,r) = 1$. And, for $1 \leq i \leq r-2$, we have,

$$P_{i,n} = \rho_i(n,r)A_0 + \rho_i(n-1,r)A_1 + \cdots + \rho_i(n-r+1,r)A_{r-1}, \tag{7}$$

for every $n \geq r$, such that $A_0 = 2^d\alpha_{r-1} + \sum_{k=1}^{r-i-2}\alpha_k + h\alpha_0$, $A_1 = h\alpha_1 + \sum_{k=2}^{r-i-1}\alpha_k, \ldots, A_{i-1} = h\alpha_{i-1} + \sum_{k=i}^{r-3}\alpha_k$, $A_i = h\alpha_i + \sum_{k=i+1}^{r-2}\alpha_k$, $A_{i+1} = h\alpha_{i+1} + \sum_{k=i+2}^{r-1}\alpha_k, \ldots, A_{r-1} = h\alpha_{r-1}$ and

$$\rho_i(n,r) = \sum_{k_0+(i+2)k_{i+1}+\cdots+rk_{r-1}=n-r} \frac{(k_0+k_{i+1}\cdots+k_{r-1})!}{k_0!k_{i+1}!\ldots k_{r-1}!}2^{dk_0}h^{k_{r-1}}, \text{ for every } n \geq r,$$

where $\rho_i(j,r) = 0$ for $j \leq r-1$ and $\rho_i(r,r) = 1$.

Proof. It is a direct application of Expressions (5) and (6) to Expression (1) for every i $(0 \leq i \leq r-2)$. □

Let consider the sequence $\mathcal{S}_j = \{\rho(n-j,r)\}_{n\geq 0}$ $(0 \leq j \leq r-1)$. Then, Theorem 2.1 shows that the set $\{\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_{r-1}\}$ is a basic fundamental system for generating the model of generalized Pell numbers (1).

Proposition 2.2. *Let the generalized Pell numbers R_n defined by Expression* (3) *with arbitrary initial conditions $\alpha_0, \ldots, \alpha_{r-1}$. Then, we have,*

$$R_{n+1} = \rho_{r-2}(n,r)A_0 + \rho_{r-2}(n-1,r)A_1 + \cdots + \rho_{r-2}(n-r+1,r)A_{r-1},$$

for every $n \geq r$, such that $A_0 = 2^d\alpha_{r-1} + h\alpha_0$, $A_1 = h\alpha_1, \ldots, A_{r-1} = h\alpha_{r-1}$ and

$$\rho_{r-2}(n,r) = \sum_{k_0+rk_{r-1}=n-r} \frac{(k_0+k_{r-1})!}{k_0!k_{r-1}!}2^{dk_0}h^{k_{r-1}}, \tag{8}$$

for every $n \geq r$, where $\rho_{r-2}(j,r) = 0$ for $j \leq r-1$ and $\rho_{r-2}(r,r) = 1$.

Proof. It is obtained by applying the Theorem 2.1 to Expression (3) for $i = r-2$. □

As mentioned before, when the initial conditions are $\alpha_0 = \cdots = \alpha_{r-2} = 0$, and $\alpha_{r-1} = 1$, we get the usual generalized Pell numbers, largely studied in the literature. In this context, the following corollary established the combinatorial expression for this important sequence of generalized Pell numbers. That is, comparing Expressions (1) and (7), we can establish the following result.

Corollary 2.3. *(Fundamental Combinatorial expression) For the sequence* (1)*, with initial conditions $\alpha_0 = \cdots = \alpha_{r-2} = 0$, and $\alpha_{r-1} = 1$, we have the combinatorial expression,*

$$P_{i,n} = \rho_i(n+1,r) = \sum_{k_0+(i+2)k_{i+1}+\cdots+rk_{r-1}=n+1-r} \frac{(k_0+k_{i+1}+\cdots+k_{r-1})}{k_0!k_{i+1}!\ldots k_{r-1}!}2^{dk_0}h^{k_{r-1}}, \tag{9}$$

for every positive integer i, and $n \geq r$, where $\rho_i(j,r) = 0$ for $j \leq r-1$ and $\rho_i(r,r) = 1$.

Expression (9) will play a fundamental role in the sequel, where the sequence $P_{i,n_{n\geq 0}}$ is considered as a fundamental solution of the difference equation (1). Moreover, we can show that Proposition 4.3 established in [16] is a particular case of Corollary 2.3 by taking $i = 0, d = h = 1$.

Consider the set $\{\{P_{i,n}^{(s)}\}_{n\geq 0},\ 1\leq s\leq r\}$ of sequences of generalized Pell numbers $P_{i,n}^{(s)}=P_{d,i,h,n}^{(s)}$ defined as follows,

$$\begin{aligned} P_{i,n+1}^{(s)} &= 2^d P_{i,n}^{(s)} + \sum_{k=i+1}^{r-2} P_{i,n-k}^{(s)} + hP_{i,n-r+1}^{(s)} \qquad \text{for} \qquad n\geq r-1, \\ P_{i,s-1}^{(s)} &= 1 \text{ and } P_{i,n}^{(s)}=0 \text{ for } 0\leq n\neq s-1\leq r-1, \end{aligned} \tag{10}$$

where in the special case $i=r-2$, we put $P_{r-2,n}^{(s)}=R_n^{(s)}$.

We call the set $\{\{P_{i,n}^{(s)}\}_{n\geq 0},\ 1\leq s\leq r\}$ as the Pell fundamental system associated with the model of generalized Pell numbers defined by (1).

The Table 1 describes the list of the first terms of set $\{\{R_n^{(s)}\}_{n\geq 0},\ 1\leq s\leq 5\}$ of the generalized Pell number of order $r=5,\ i=3, d=1$ and $h=5$. The sequences of generalized Pell numbers are defined as follows,

$$\begin{aligned} R_{n+1}^{(s)} &= 2R_n^{(s)}+5R_{n-4}^{(s)} \qquad \text{for} \qquad n\geq 4, \\ R_{s-1}^{(s)} &= 1 \text{ and } R_n^{(s)}=0 \text{ for } 0\leq n\neq s-1\leq 4. \end{aligned}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	...
$R_n^{(1)}$	1	0	0	0	0	5	10	20	40	80	185	420	...
$R_n^{(2)}$	0	1	0	0	0	0	5	10	20	40	80	185	...
$R_n^{(3)}$	0	0	1	0	0	0	0	5	10	20	40	80	...
$R_n^{(4)}$	0	0	0	1	0	0	0	0	5	10	20	40	...
$R_n^{(5)}$	0	0	0	0	1	2	4	8	16	37	84	188	...

Table 1 : First terms of set$\{\{R_n^{(s)}\}_{n\geq 0},\ 1\leq s\leq 5\}$

The Table 2 describes the list of the first terms of set $\{\{P_{2,2,3,n}^{(s)}\}_{n\geq 0},\ 1\leq s\leq 5\}$ of the generalized Pell number of order $r=5,\ i=2,\ d=2$ and $h=3$. The sequences of generalized Pell numbers are defined as follows,

$$\begin{aligned} P_{2,2,3,n}^{(s)} &= 2^2P_{2,2,3,n-1}^{(s)}+P_{2,2,3,n-4}^{(s)}+3P_{2,2,3,n-5}^{(s)} \qquad \text{for} \qquad n\geq 5, \\ P_{2,2,3,s-1}^{(s)} &= 1 \text{ and } P_{2,2,3,n}^{(s)}=0 \text{ for } 0\leq n\neq s-1\leq 4. \end{aligned}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	...
$P_{2,2,3,n}^{(1)}$	1	0	0	0	0	3	12	48	192	771	3105	12504	...
$P_{2,2,3,n}^{(2)}$	0	1	0	0	0	1	7	28	112	449	1806	7273	...
$P_{2,2,3,n}^{(3)}$	0	0	1	0	0	0	1	7	28	112	449	1806	...
$P_{2,2,3,n}^{(4)}$	0	0	0	1	0	0	0	1	7	28	112	449	...
$P_{2,2,3,n}^{(5)}$	0	0	0	0	1	4	16	64	257	1035	4168	16784	...

Table 2 : First terms of set$\{\{P_{2,2,3,n}^{(s)}\}_{n\geq 0},\ 1\leq s\leq 5\}$

Remark 2.4. *As mentioned before in the Introduction, for $d=0$, Equation (2) is none other than the one that defines the generalized Fibonacci numbers. Therefore, results of this Section are still valid for the generalized Fibonacci sequences, especially, Theorem 2.1 and Corollary 2.3. For the* *recurrence given by Expression* (2) *with* $d = 0$, *it was proved in [Proposition 3.3, Proposition 3.4, [15]] the identities,*

$$\begin{aligned} P_n^{(s)} &= \sum_{j=1}^{s} \rho(n+s-j,r), \quad \textit{for } n \geq r+s, \textit{ when } 2 \leq s \leq r, \\ P_n^{(1)} &= P_{n-1}^{(r)} = \rho(n,r), \textit{ for } n \geq r+1, \end{aligned}$$

where the $\rho(n,r)$ *are given as in* (6) *with* $a_0 = 1$, $a_1 = \cdots = a_{r-1} = 1$. *As an analogous result, for recurrence given by Expression* (2) *with* $d = 1$, *it was proved in [Proposition 4.4, [16]] the identities,*

$$\begin{aligned} P_n^{(s)} &= \sum_{j=1}^{s} \rho(n+s-j,r), \quad \textit{for } n \geq r+s, \textit{ when } 2 \leq s \leq r, \\ P_n^{(1)} &= P_{n-1}^{(r)} = \rho(n,r), \textit{ for } n \geq r+1, \end{aligned}$$

where the $\rho(n,r)$ *are given as in* (6) *with* $a_0 = 2$, $a_1 = \cdots = a_{r-1} = 1$.

The application of Theorem 2.1 implies the more general propositions bellow.

Proposition 2.5. *Fixed positive integers* r *and* i, *where* $1 \leq i \leq r-3$, *let* $\{\{P_{i,n}^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$ *be the Pell fundamental system defined as in* (10). *The combinatorial expression of each element* $P_{i,n}^{(s)}$, *where* $1 \leq s \leq r$, *is given by,*

$$P_{i,n}^{(s)} = \sum_{j=s-r+i+1}^{s-2} \rho_i(n-j,r) + h\rho_i(n-s+1,r), \quad \textit{when } 2 \leq s \leq r, \tag{11}$$

$$P_{i,n}^{(1)} = hP_{i,n-1}^{(r)} = hP_{i,n-1} = h\rho_i(n,r), \textit{ for } n \geq r+1, \tag{12}$$

with $n \geq r+s$, *where the* $\rho_i(n,r)$ *are given as in* (6) *such that* $a_0 = 2^d$, $a_1 = ... = a_i = 0, a_{i+1} = ... = a_{r-1} = 1$.

Moreover, for $i = r-2$ we can establish the combinatorial expressions of the sequences of generalized Pell numbers $\{R_n\}_{n\geq 0}$ defined as in (3). Indeed, we obtain the result analogous to Proposition 2.5.

Proposition 2.6. *For* $i = r-2$, *let consider the Pell fundamental system* $\{\{R_n^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$ *defined as in* (10). *Then, the combinatorial expression of each element* $\{R_n^{(s)}\}_{n\geq 0}$, *where* $1 \leq s \leq r$, *is given by,*

$$R_n^{(s)} = h\rho_2(n-s+1,r), \quad \textit{when } 2 \leq s \leq r-1, \tag{13}$$

$$R_n^{(1)} = hR_{n-1}^{(r)} = hR_{n-1} = h\rho_2(n+1,r), \textit{ for } n \geq r+1, \tag{14}$$

with $n \geq r+s$, *where the* $\rho_2(n,r)$ *are given as in* (6) *such that* $a_0 = 2^d$, $a_1 = ... = a_{r-2} = 0, a_{r-1} = 1$.

Remark 2.7. *As mentioned before in the Introduction, for* $d = 0$ *and* $h = 1$, *Equation* (3) *defines Fibonacci* $r-$*numbers. Therefore, results of this Section are still valid for this sequence of numbers, especially, Proposition 2.6.*

Let $\{W_{i,n}\}_{n\geq 0}$ be a sequence of generalized Pell numbers (1), with arbitrary initial conditions α_0, α_1, ..., α_{r-1}. Let $\{\tilde{W}_{i,n}\}_{n\geq 0}$ be the sequence defined by,

$$\tilde{W}_{i,n} = \alpha_0 P_{i,n}^{(1)} + \alpha_1 P_{i,n}^{(2)} + ... + \alpha_{r-1} P_{i,n}^{(r)} \text{ for every } n \geq 0.$$

We show that $\tilde{W}_{i,0} = \alpha_0$, $\tilde{W}_{i,1} = \alpha_1$, ..., $\tilde{W}_{i,r-1} = \alpha_{r-1}$. On the other hand, the general term of the sequence $\{\tilde{W}_{i,n}\}_{n\geq 0}$ satisfies the recursive relation (1). Therefore, for every $n \geq 0$, we have $W_{i,n} = \tilde{W}_{i,n}$. Especially, when $d = h = 1, \alpha_0 = ... = \alpha_i = 0$ and $\alpha_{i+1} = ... = \alpha_{r-1} = 1$, we get the associated sequence of generalized Pell numbers $\{\widetilde{R}_n\}_{n\geq 0}$, defined by the recursive relation (3) and called the generalized Pell (r, i)-numbers (see [12]).

Proposition 2.8. *Let $\{W_{i,n}\}_{n\geq 0}$ be a sequence of generalized Pell numbers* (1), *with arbitrary initial conditions α_0, α_1, ..., α_{r-1}. Then, for every $n \geq 0$, we have,*

$$W_{i,n} = \alpha_0 P_{i,n}^{(1)} + \alpha_1 P_{i,n}^{(2)} + ... + \alpha_{r-1} P_{i,n}^{(r)},$$

Especially, for the generalized Pell (r, i)-numbers $\widetilde{R}_n$, we have

$$\widetilde{R}_n = R_n^{(i+1)} + ... + R_n^{(r)}.$$

In other terms, the set $\{\{P_{i,n}^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$ is a basis of the vector space $\mathcal{E}_{\mathbb{K}}^{(i,r)}$ (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) of solutions of Equation (1) considered as a difference equation.

The Proposition 2.8 is general case of Proposition 2.1 in [16]. Since the generalized Pell is also linked to the generalized Pell (r, i)-numbers considered in [12], we deduce from Proposition 2.8 and Expressions (13)-(14), a combinatorial expression of the generalized Pell (r, i)-numbers, namely,

$$\widetilde{R}_n = \sum_{j=i+1}^{r} \rho_2(n - j + 1, r),$$

for every $n \geq 0$, where the $\rho_2(n, r)$ are given as in (6) such that $a_0 = 2$, $a_1 = ... = a_{r-2} = 0, a_{r-1} = 1$.

3. Some Identities and Combinatorial Identities for the Model of Generalized Pell numbers

In this section we are interested in some generalized Pell numbers identities and the related combinatorial identities. To this aim, let us proceed as in [15] by considering the notion of the generalized Pell fundamental system. Let consider the vector column $P(i, j, n) = (P_{i,n}^{(j)};\ P_{i,n+1}^{(j)};\ \dots;\ P_{i,n+r-1}^{(j)})^t$, for every $n \geq r - 1$, and j $(1 \leq j \leq r)$, and the matrix,

$$\widehat{C}_{\mathfrak{P}}(i, n) = [P(i, 1, n), \dots, P(i, j, n), \dots, P(i, r, n)],$$

Let $\mathcal{S}_j = \{P_{j,n}^{(s)}\}_{n\geq 0}$ $(0 \leq j \leq r - 1)$ and consider the basic set $\mathfrak{S}_r = \{\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_{r-1}\}$, called the Pell fundamental system, related to the model generalized Pell numbers (1). Then, the matrix $\widehat{C}_{\mathfrak{P}}(i, n) = (c_{kj}^{(n)})_{1\leq k,\, j\leq r}$, represents the Pell Casoratian matrix associated with $\mathfrak{S}_r$.

The main goal here, is to exhibit the explicit expressions for the entries $c_{kj}^{(n)}$ of the matrix $\widehat{C}_{\mathfrak{P}}(i,n)$, and derive some identities, related to the model of generalized Pell numbers (1). A direct verification shows that the Casoratian matrix can be written under the form,

$$\widehat{C}(n) = J \times \mathbb{M}_n \times J,$$

where $J = (b_{k,j})_{1\leq k,j\leq r}$ is the anti-diagonal unit matrix, namely, $b_{k,j} = 1$, for $k+j = r+1$, and $b_{k,j} = 0$, otherwise and $\mathbb{M}_n = (P_{i,n+r-k-1}^{(j)})_{1\leq k,j\leq r}$. We show that the matrix $\mathbb{M}_{i,n}$, can be written under the form $\mathbb{M}_{i,n} = \mathbb{A}_i^n$, where $\mathbb{A}_i$ is the classical companion matrix,

$$\mathbb{A}_i = \mathbb{A}[2^d, 0, \cdots, 1, \cdots, 1, h] = \begin{pmatrix} 2^d & 0 & \cdots & 1 & \cdots & 1 & h \\ 1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(for more details see, [2] and references therein). Hence, we get the following property.

Proposition 3.1. *Consider the Pell fundamental system $\{\{P_{i,n}^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$ defined as in* (10)*. Then, the associated Casoratian matrix $\widehat{C}(n)$ and the powers $\mathbb{A}_i^n$ of the companion matrix $\mathbb{A}_i$ are similar. More precisely, we have the matrix identity,*

$$\widehat{C}(n) = J\mathbb{A}_i^n J = (c_{kj}^{(n)})_{1\leq k,j\leq r}, \tag{15}$$

for every $n \geq 0$, where the entries $c_{kj}^{(n)}$ are given by $c_{kj}^{(n)} = P_{i,n+i-1}^{(j)}$ $(1 \leq k, j \leq r)$, and $J = (b_{k,j})_{1\leq k,j\leq r}$ is the anti-diagonal unit matrix.

Expression (15) implies the matrix identity $\widehat{C}(n+m) = \widehat{C}(n).\widehat{C}(m)$, for every n and m. Hence, the entries of the matrix $\widehat{C}(n+m) = (c_{kj}^{(n+m)})_{1\leq k,j\leq r}$, are expressed in terms of those of the matrices $\widehat{C}(m) = (c_{kj}^{(m)})_{1\leq i,j\leq r}$ and $\widehat{C}(n) = (c_{ij}^{(n)})_{1\leq i,j\leq r}$ as follows,

$$c_{kj}^{(n+m)} = \sum_{l=1}^{r} c_{kl}^{(n)} c_{lj}^{(m)} = \sum_{l=1}^{r} c_{kl}^{(m)} c_{lj}^{(n)}, \text{ for every } n,\ m \geq 0, \tag{16}$$

where $1 \leq k,\ j \leq r$. In fact, according to Proposition 3.1 and Expression (16), we get the following identities for the model of generalized Pell numbers $P_{i,n}^{(s)}$.

Proposition 3.2. *(Identities for the model of generalized Pell numbers) Consider the Pell fundamental system $\{\{P_{i,n}^{(s)}\}_{n\geq 0};\ 1 \leq s \leq r\}$ defined as in* (10)*. Then, is valid the following identity,*

$$P_{i,m+s+p}^{(q)} = \sum_{d=1}^{r} P_{i,m+p}^{(d)} P_{i,s+d-1}^{(q)} = \sum_{d=1}^{r} P_{i,s+p}^{(d)} P_{i,m+d-1}^{(q)},$$

for any integer m , $s \geq 0$ and p, q $(1 \leq p,\ q \leq r)$. Specially for $q = r$ we have the identity,

$$P_{i,m+s+p}^{(r)} = P_{i,m+s+p} = \sum_{d=1}^{r} P_{i,m+p}^{(d)} P_{s+d-1} = \sum_{d=1}^{r} P_{i,s+p}^{(d)} P_{i,m+d-1},$$

where $P^{(1)}_{i,n+1} = P_{i,n} = \rho_1(n+1,r)$ *and* $P^{(s)}_{i,n} = \sum_{j=s-r+i+1}^{s-2} P_{i,n-j-1} + hP_{i,n-s}$. *And more generally, for* $1 \leq q \leq r-1$, *we have,*

$$P^{(q)}_{i,m+s+p} = \sum_{d=1}^{r} \left[\sum_{j=d-r+i+1}^{d-2} \delta_{m,p} \right] \left[\sum_{j=q-r+i+1}^{q-2} \delta_{s,q} \right],$$

where $\delta_{m,p} = P_{i,m+p-j-1} + hP_{i,m+p-d}$ *and* $\delta_{s,q} = P_{i,s+d-j-2} + hP_{i,s+d-1-q}$.

Combining the identities of Proposition 3.2 with Theorem 2.1, Corollary 2.3 and Proposition 2.5, we can establish some combinatorial identities, involving the expressions of $\rho(n,j)$ and $\rho(n,r)$. More precisely, identities of Proposition 3.2 and Expression (9) applied to Expressions (11)-(12) and Expressions (13)-(14), we arrive at the following combinatorial identities.

Proposition 3.3. *(Combinatorial identities for the model of generalized Pell numbers) The combinatorial expressions of the generalized Pell numbers identity* (11), *is given by*

$$\rho_i(m+s+1,r) = \sum_{d=1}^{r} \left[\sum_{j=d-r+i+1}^{d-2} \Delta_{m,s} \right] \rho_i(s+d,r),$$

where $\Delta_{m,s} = \rho_i(m-j,r) + h\rho_i(m-s+1,r)$, *and*

$$\sum_{\mathcal{S}_1} \rho_i((m+s+p)-j,r) + h\rho_i((m+s+p)-q+1,r) = \sum_{d=1}^{r} \left(\sum_{\mathcal{S}_2} \Delta_{m+p,d} \right) \left(\sum_{\mathcal{S}_3} \Delta_{s+d,q} \right),$$

where $\mathcal{S}_1 = \{j,\ \text{with } q-r+i+1 \leq j \leq q-2\}$, $\mathcal{S}_2 = \{j,\ \text{with } d-r+i+1 \leq j \leq d-2\}$, $\mathcal{S}_3 = \{j,\ \text{with } q-r+i+1 \leq j \leq s-2\}$, *and* $\Delta_{m+p,d} = \rho_i(m+p-j,r) + h\rho_i(m+p-d+1,r)$, $\Delta_{s+d,q} = \rho_i(s+d-1-j,r) + h\rho_i(s+d-1-q+1,r)$.

Propositions 3.2 and 3.3 represent a generalization of the identities linked to the Fibonacci number established in [Proposition 3.3, Proposition 3.4, Corollary 3.5, [15]] and Pell number established in [Theorem 3.1, Proposition 4.2, Corollary 4.0, [16]]. Specially for Pell numbers, with parameters $d = 1$ and $h = 1$ in the model of generalized Pell numbers (1), namely the expression

$$P_{1,0,1,n+1} = 2P_{1,0,1,n} + P_{1,0,1,n-1} + \cdots + P_{1,0,1,n-r+1}, \tag{17}$$

for $n \geq r$, we have the result.

Corollary 3.4 (Corollary 4.0, [16]). *Consider the Pell fundamental system* $\{\{P^{(s)}_n\}_{n\geq 0};\ 1 \leq s \leq r\}$ *associated with the sequences of generalized Pell numbers* (17). *Then, for every* m, $s \geq 0$, q *(*$1 \leq q \leq r$*), we have the following combinatorial identities,*

$$\rho_0(m+s+1,r) = \sum_{d=1}^{r} \left[\sum_{j=1}^{d} \rho_0(m-j+1,r) \right] \rho_0(s+d,r),$$

$$\sum_{k=1}^{q} \rho_0(n+s-k+1,r) = \sum_{d=1}^{r} \left[\sum_{1\leq i\leq d,\ 1\leq j\leq q} \rho_0(n-i+1,r)\rho_0(s+d-j,r) \right].$$

4. ANALYTIC REPRESENTATION OF PELL NUMBERS (2)-(3)

This section is devoted to the study of the analytic expression of a large class of the model generalized Pell numbers (1), without the use of the determinant techniques. Namely, we are interested in the analytic expressions of the generalized Pell numbers (2) and the generalized Pell numbers (3).

Recall that for linear recursive sequence of Fibonacci type (4), the analytic expression is expressed in terms of the roots of the associated so-called characteristic polynomial and their multiplicities (see, instance, [3,8,18]). For the class of Pell numbers (2)-(3), the characteristic polynomials are given as follows,

$$P(z) = z^r - 2^d z^{r-1} - z^{r-2} - \ldots - z - 1 \text{ and } R(z) = z^r - 2^d z^{r-1} - 1,$$

where $d \geq 0$.

4.1. Study of the case $d \geq 1$.

Lemma 4.1. *The roots of the polynomial*

$$P(z) = z^r - 2^d z^{r-1} - z^{r-2} - ... - z - 1,$$

are simple.

Proof. Consider $d \geq 1$. Then, the polynomial $P(z) = z^r - 2^d z^{r-1} - z^{r-2} - ... - z - 1$ can be written under the form,

$$P(z) = z^r - (2^d - 1)z^{r-1} - (z^{r-1} + ... + z + 1) = z^r - (2^d - 1)z^{r-1} - \frac{z^r - 1}{z - 1}.$$

Since $P(1) \neq 0$, we have $P(z) = z^r - (2^d - 1)z^{r-1} - \frac{z^r - 1}{z-1} = \frac{S(z)}{z-1}$, where

$$S(z) = z^{r+1} - (2^{d+1} + 1)z^r + (2^d - 1)z^{r-1} + 1.$$

Let $Z(P) = \{z \in \mathbb{C}, P(z) = 0\}$ and $\lambda \in Z(P)$. Since $P(1) \neq 0$, we show easily that $P(\lambda) = 0$ if, and only if, $S(\lambda) = 0$, or equivalently,

$$\lambda^{r+1} - (2^d + 1)\lambda^r + (2^d - 1)\lambda^{r-1} + 1 = 0. \tag{18}$$

Suppose that λ is a root of $P(z)$, with multiplicity $m \geq 2$. Since $S(\lambda) = 0$, $\lambda \neq 1$ and $P'(z) = \frac{S(z) - S'(z)(z-1)}{(z-1)^2}$, where $P'(z)$ denote the derivative of $P(z)$, we derive $S'(\lambda) = 0$, namely, we have the

$$S'(\lambda) = [(r + 1)\lambda^2 - (2^d + 1)r\lambda + (2^d - 1)(r - 1)]\lambda^{r-2} = 0. \tag{19}$$

Since $P(0) = -1 \neq 0$, or equivalently $0 \notin Z(P)$, we derive the following equation,

$$(r + 1)\lambda^2 - (2^d + 1)r\lambda + (2^d - 1)(r - 1) = 0, \tag{20}$$

whose roots are given by,

$$\lambda_1 = \frac{(2^d + 1)r + \sqrt{\Delta}}{2(r + 1)} \text{ and } \lambda_2 = \frac{(2^d + 1)r - \sqrt{\Delta}}{2(r + 1)},$$

where $\Delta = (2^d + 1)^2 r^2 - 4(2^d - 1)(r^2 - 1)$.

On the other side, taking into account Expressions (18) and (19), we derive the following equation,

$$(r+1)\lambda^{r+1}-(2^d+1)r\lambda^r+(2^d-1)(r-1)\lambda^{r-1}$$
$$=r[\lambda^{r+1}-(2^d+1)\lambda^r+(2^d-1)\lambda^{r-1}+1]+\lambda^{r+1}-(2^d-1)\lambda^{r-1}-r=0.$$

Using Equation (18), we derive $\lambda^{r+1}-(2^d-1)\lambda^{r-1}-r=0$, hence we have the following identity,

$$\lambda^{r+1}-(2^d-1)\lambda^{r-1}=\lambda^{r-1}[\lambda^2-(2^d-1)]=r. \tag{21}$$

Case 1. Let now establish that the root λ_1 is not a root of $P(z)$ of multiplicity ≥ 2. The expression of λ_1, takes the form, $\lambda_1=\frac{(2^d+1)r+\sqrt{\Delta}}{2(r+1)}=\frac{(2^d+1)(r+1)+\sqrt{\Delta}-(2^d+1)}{2(r+1)}$, which implies $\lambda_1=\dfrac{(2^d+1)}{2}+\dfrac{\sqrt{\Delta}-(2^d+1)}{2(r+1)}$. Since

$$\Delta=(2^d+1)^2r^2-4(2^d-1)(r^2-1)=[2^d(2^d-2)+5]r^2+4(2^d-1), \tag{22}$$

a straightforward computation shows that,

$$\frac{\sqrt{\Delta}-(2^d+1)}{2(r+1)} = \frac{(2^d(2^d-2)+5)(r^2-1)}{2(r+1)(\sqrt{\Delta}+(2^d+1))}>0.$$

Since $r\geq 3$ and $d\geq 1$ and $\frac{\sqrt{\Delta}-(2^d+1)}{2(r+1)}>0$, then $\lambda_1>\frac{(2^d+1)}{2}>\frac{3}{2}$. On the other side, we have,

$$\lambda_1^2-(2^d-1)=\frac{A(r,d)}{4(r+1)^2},$$

where $A(r,d)=((2^d+1)r)^2+2(2^d+1)r\sqrt{\Delta}+\Delta-4(2^d-1)(r+1)^2$, let analyze the sign of the expression $\Omega(r,d)=A(r,d)-4(r+1)^2$. We have

$$\begin{aligned}\Omega(r,d) &= ((2^d+1)r)^2+2(2^d+1)r\sqrt{\Delta}+\Delta-4(2^d-1)(r+1)^2-4(r+1)^2\\ &= ((2^{2d}-2^{d+1})r^2+r^2+(2^{d+1})r(\sqrt{\Delta}-4)+2r\sqrt{\Delta}+\Delta-2^{d+2}.\end{aligned}$$

Since $r\geq 3, d\geq 1$, Expression (22) allows us to derive that $\Delta-2^{d+2}>0$ and $\sqrt{\Delta}-4>0$, because $\Delta>49$. Therefore, we obtain $\Omega(r,d)=A(r,d)-4(r+1)^2>0$. Hence, we have $A(r,d)>4(r+1)^2$, thus $\lambda_1^2-(2^d-1)>1$.

Finally, since $\lambda_1>\frac{3}{2}$ we show that $\lambda_1^{r-1}>(\frac{3}{2})^{r-1}>r$, for $r\geq 3$. Therefore, Expression (21), shows that, $r=\lambda_1^{r-1}[\lambda_1^2-(2^d-1)]>r$, which is impossible. Conclusion, λ_1 is not a root of the polynomial $P(z)$ of multiplicity ≥ 2.

Case 2. Let us also establish that $\lambda_2=\dfrac{(2^d+1)r-\sqrt{\Delta}}{2(r+1)}$ is not a root of $P(z)$ of multiplicity ≥ 2. By using Expression (22), we have,

$$\lambda_2 = \frac{((2^d+1)r-\sqrt{\Delta})((2^d+1)r+\sqrt{\Delta})}{2(r+1)((2^d+1)r+\sqrt{\Delta})}=\frac{2(r-1)(2^d-1)}{(2^d+1)r+\sqrt{\Delta}}>0,$$

and

$$\lambda_2^2-(2^2-1)=\left[\frac{4(r-1)^2(2^d-1)}{[(2^d+1)r+\sqrt{\Delta}]^2}-1\right](2^d-1).$$

On the other hand, a straightforward computation shows that,

$$[(2^d+1)r+\sqrt{\Delta}]^2-4(r-1)^2(2^d-1)=(2^{2d}-2^{d+1}+4)r^2+4(2^d-1)(r-2)+2(2^2+1)r\sqrt{\Delta}+\Delta>0,$$

which implies that $[(2^d+1)r+\sqrt{\Delta}]^2 > 4(r-1)^2(2^d-1)$ or equivalently $\frac{4(r-1)^2(2^d-1)}{[(2^d+1)r+\sqrt{\Delta}]^2} < 1$. Therefore, we have,

$$\lambda_2^2 - (2^2-1) = \left[\frac{4(r-1)^2(2^d-1)}{[(2^d+1)r+\sqrt{\Delta}]^2} - 1\right](2^d-1) < 0.$$

Once again, taking into account Expression (21), namely, $\lambda_2^{r-1}[\lambda_2^2 - (2^d-1)] = r$, we derive,

$$\lambda_2^{r-1}[\lambda_2^2 - (2^d-1)] = r < 0$$

which is impossible. Consequently, the root λ_2 of Equation (20) is not a root of the polynomial $P'(z)$ or equivalently, λ_2 is not a root of multiplicity ≥ 2 of the polynomial $P(z)$. Therefore, the roots of the polynomial $P(z)$ are simple. $\square$

Remark 4.2. *Note that for $d = 1$ we recover Lemma 5.1 of [16], namely, Lemma 4.1 is a generalization of [Lemma 5.1, [16]].*

For the characteristic the polynomial $R(z) = z^r - 2^d z^{r-1} - h$ of the generalized Pell numbers (3), we have the following result.

Lemma 4.3. *For integers $d \geq 1$ and $h \geq 1$, the roots of the polynomial,*

$$R(z) = z^r - 2^d z^{r-1} - h,$$

are simple.

Proof. For $r = 2$ we have $R(z) = z^2 - 2^d z - h$, and since $\Delta = (-2)^2 d + 4k > 0$ the roots of the polynomial $R(z)$ are simple. For $r \geq 3$, if λ is a root of $R(z)$, we have,

$$\lambda^r - 2^d\lambda^{r-1} - h = 0 \Leftrightarrow \lambda^{r-1}(\lambda - 2^d) = h. \tag{23}$$

Suppose that λ is a root of multiplicity $m \geq 2$, then $R'(\lambda) = 0$. Therefore, we have,

$$r\lambda^{r-1} - 2^d(r-1)\lambda^{r-2} = 0 \Leftrightarrow \lambda^{r-2}(r\lambda - 2^d(r-1)) = 0.$$

Since $R(0) \neq 0$, we derive that $\lambda = \dfrac{2^d(r-1)}{r}$. And using Expression (23), we show

$$h = \left(\frac{2^d(r-1)}{r}\right)^{r-1}\left(\frac{2^d(r-1)}{r} - 2^d\right) = \left(\frac{2(r-1)}{r}\right)^{r-1}\left(-\frac{2^d}{r}\right),$$

which is impossible, since h is a positive integer. Therefore, the roots of the polynomial $R(z) = z^r - 2^d z^{r-1} - h$ are simple. $\square$

Let apply Lemmas 4.1-4.3 for providing the analytic formula of generalized Pell numbers (2)-(3). The process present in these two lemmas allows us to avoid the heavy techniques of the determinant by considering the result [3, Theorem 2.2]. That is, the combinatorial expression of $\rho(n,r)$ related to the general case of linear difference equation (4) is expressed in terms of the roots of the polynomial $P(z) = z^r - a_0 z^{r-1} - \ldots - a_{r-2}z - a_{r-1}$. More precisely, the sequence $\{\rho(n,r)\}_{n\geq 0}$ defined by (6), is a linear recursive of type (4), and its analytical expression is given in the following lemma.

Lemma 4.4. *(see [1,3]) Let $\{\rho(n,r)\}_{n\geq 0}$ be the sequence defined by* (6). *Suppose that the roots $\lambda_1, \cdots, \lambda_r$ of its characteristic polynomial $P(z) = z^r - a_0 z^{r-1} - \ldots - a_{r-2}z - a_{r-1}$ $(a_{r-1} \neq 0)$ satisfy $\lambda_i \neq \lambda_j$ for $i \neq j$. Then, we have*

$$\rho(n,r) = \sum_{i=1}^{r} \frac{\lambda_i^{n-1}}{P'(\lambda_i)} = \sum_{i=1}^{r} \frac{\lambda_i^{n-1}}{\prod_{k\neq i}(\lambda_i - \lambda_k)} \text{ for every } n \geq r,$$

otherwise $\rho(r,r) = 1$, $\rho(i,r) = 0$ for $i \leq r-1$, where $P'(z) = \frac{dP}{dz}(z)$.

Following Corollary 2.3 and Proposition 2.5 the combinatorial expression of the sequences $\{P_{i,n}\}_{n\geq 0}$ of the Pell fundamental system (10) are formulated in terms of the $\rho(n,r)$, given by Expressions (9), (11) and (12), namely, we have,

$$P_{i,n} = \rho_i(n+1,r),\ P_{i,n}^{(1)} = hP_{i,n-1}^{(r)} = hP_{i,n-1} = h\rho_i(n,r),$$

$$P_{i,n}^{(s)} = \sum_{j=s-r+i+1}^{s-2} \rho_i(n-j,r) + h\rho_i(n-s+1,r),\ ,$$

for $n \geq r$, $n \geq r+1$ or $r \geq r+s$, respectively, and where the $\rho_i(n,r)$ are given as in (6) such that $a_0 = 2^d$, $a_1 = \ldots = a_i = 0, a_{i+1} = \ldots = a_{r-1} = 1$, namely,

$$\rho_i(n,r) = \sum_{k_0+(i+2)k_{i+1}+\cdots+rk_{r-1}=n-r} \frac{(k_0 + k_{i+1} + \cdots + k_{r-1})!}{k_0! k_{i+1}! \ldots k_{r-1}!} 2^{dk_0}, \text{ for } n \geq r.$$

where $\rho_i(r,r) = 1$ and $\rho_i(n,r) = 0$ for $n \leq r-1$.

Now using Lemmas 4.1-4.4 and Expressions (11)-(12), we can formulate the analytical expressions of the family of Pell numbers constituting the Pell fundamental system defined as in (10), as follows.

Theorem 4.5. *Let $h = 1$ and $\{\{P_n^{(s)}\}_{n\geq 0},\ 1 \leq s \leq r\}$ be the Pell fundamental system defined as in* (10). *Then, the analytic expression of each $P_n^{(s)}$, $(1 \leq s \leq r)$ is given by,*

$$P_n = \rho_0(n+1,r) = \sum_{j=1}^{r} \frac{\lambda_j^{n-1}}{P'(\lambda_j)} = \sum_{j=1}^{r} \frac{\lambda_j^{n-1}}{\prod_{k\neq j}(\lambda_j - \lambda_k)}, \text{for } n \geq r,$$

$$P_n^{(1)} = P_{n-1} = \rho_0(n,r) = \sum_{j=1}^{r} \frac{\lambda_j^{n-2}}{P'(\lambda_j)} = \sum_{j=1}^{r} \frac{\lambda_j^{n-2}}{\prod_{k\neq j}(\lambda_j - \lambda_k)}, \text{for } n \geq r+1,$$

$$P_n^{(s)} = \sum_{j=s-r+1}^{s-2} \rho_0(n-j,r) + \rho_0(n-s+1,r) = \sum_{j=s-r+1}^{s-2} \sum_{t=1}^{r} \frac{\lambda_t^{n-j-2}}{P'(\lambda_t)} + \frac{\lambda_t^{n-s-1}}{P'(\lambda_t)}$$

where $\lambda_1, \cdots, \lambda_r$ the simple roots of the polynomial $P(z) = z^r - 2^d z^{r-1} - z^{r-2} - \ldots - z - 1$.

We can show that for $d = 1$, Theorem 4.5 is nothing else but the Theorem 5.1 established in [16]. We illustrate Theorem 4.5 by considering the following numerical cases.

Example 4.6. *For $r = 3$ and $d = h = 1$, we have the fundamental system $\{\{P_n^{(s)}\}_{n\geq 0},\ 1 \leq s \leq 3\}$. In this case, the roots of the characteristic polynomial associated $P(z) = z^3 - 2z^2 - z - 1$ are $\lambda_1 \approx 2.5468$, $\lambda_2 \approx -0.27341 - 0.56382i$ and $\lambda_3 \approx -0.27341 + 0.56382i$. Then, applying the Theorem 4.5 we obtain*

$$P_n=\frac{\lambda_1^{n-1}}{3\lambda_1^2-4\lambda_1-1}+\frac{\lambda_2^{n-1}}{3\lambda_2^2-4\lambda_2-1}+\frac{\lambda_3^{n-1}}{3\lambda_3^2-4\lambda_3-1}, \textit{for } n\geq 3,$$

$$P_n^{(1)}=\frac{\lambda_1^{n-2}}{3\lambda_1^2-4\lambda_1-1}+\frac{\lambda_2^{n-2}}{3\lambda_2^2-4\lambda_2-1}+\frac{\lambda_3^{n-2}}{3\lambda_3^2-4\lambda_3-1}, \textit{for } n\geq 4,$$

$$P_n^{(2)}=\frac{\lambda_1^{n-2}+\lambda_1^{n-3}}{3\lambda_1^2-4\lambda_1-1}+\frac{\lambda_2^{n-2}+\lambda_2^{n-3}}{3\lambda_2^2-4\lambda_2-1}+\frac{\lambda_3^{n-2}+\lambda_3^{n-3}}{3\lambda_3^2-4\lambda_3-1}, \textit{for } n\geq 5.$$

The combination of Proposition 2.8 and Theorem 4.5 permit us to get the analytical expression for the general setting of any sequence of generalized Pell numbers, with arbitrary initial conditions without solving the associated Vandermonde linear system, namely, we have the following result.

Proposition 4.7. *Let consider the sequence $\{W_n\}_{n\geq 0}$ defined by, $W_n=\alpha_0 P_n^{(1)}+\alpha_1 P_n^{(2)}+\ldots+\alpha_{r-1}P_n^{(r)}$, for every $n\geq 0$, where $h=1$ and $\{\{P_n^{(s)}\}_{n\geq 0},\ 1\leq s\leq r\}$ is the Pell fundamental system defined as in* (10). *Then, the analytical expression to W_n is given by $W_n=\sum_{k=1}^{r}\sum_{j=1}^{r}\frac{\alpha_{k-1}\lambda_j^{n-2}}{\prod_{k\neq j}(\lambda_j-\lambda_k)}$. where $\lambda_1,\cdots,\lambda_r$ the simple roots of the polynomial $P(z)=z^r-2^d z^{r-1}-z^{r-2}-\ldots-z-1$.*

Similarly, Lemmas 4.3-4.4 and Expressions (13)-(14) allow us formulate the following analogous results of the family of Pell numbers constituting the Pell fundamental systems related to Expression (3) as follows.

Theorem 4.8. *Let $\mathfrak{R}_r=\{\{R_n^{(s)}\}_{n\geq 0},\ 1\leq s\leq r\}$ be the Pell fundamental system related to generalized Pell numbers given by Expression* (3). *Then, the analytical expression of each $\{R_n^{(s)}\}_{n\geq 0},\ 1\leq s\leq r\}$ is given by,*

$$R_n=\rho_2(n+1,r)=\sum_{i=1}^{r}\frac{\lambda_i^{n-1}}{R'(\lambda_i)}, \textit{ for } n\geq 1,$$

$$R_n^{(1)}=hR_{n-1}=h\rho_2(n,r)=h\sum_{i=1}^{r}\frac{\lambda_i^{n-2}}{R'(\lambda_i)}, \textit{ for } n\geq r$$

$$R_n^{(s)}=h\rho_2(n-s+1,r)=h\sum_{i=1}^{r}\frac{\lambda_i^{n-s-1}}{R'(\lambda_i)}, \textit{ for } n\geq r+s.$$

where $\lambda_1,\cdots,\lambda_r$ the simple roots of the polynomial $R(z)=z^r-2^d z^{r-1}-h$.

In best of our knowledge the presented analytical representations are not current in the literature. The analytic formula for generalized Pell $(r,r)-$ numbers is presented in terms of determinant in [Theorem 6 and Corollary 7, [12]]. Under the precedent discussion and notation this analytic formula, only in terms of powers of roots of characteristic polynomial associated to the generalized Pell $(r,r)-$ numbers is given by the first representation in 4.8.

Since the Pell fundamental system associated with generalized Pell numbers (1) is also linked to the generalized Pell (r,i)-numbers, we deduce from Proposition 2.8 and Theorem 4.8, an analytical expression of the generalized Pell (r,i)-numbers without using determinant.

Proposition 4.9. *Let the set of the generalized Pell (r,i)-numbers $\widetilde{R}_n$. Then, the analytic expression of each $\widetilde{R}_n$, for $n \geq r+i$ is given by,*

$$\widetilde{R}_n = \sum_{k=1}^{i+2} \sum_{j=1}^{r} \frac{\lambda_k^{n-k}}{R'(\lambda_j)}, \tag{24}$$

where $\lambda_1, \cdots, \lambda_r$ the simple roots of the polynomial $R(z) = z^r - 2z^{r-1} - 1$ and $R'(z) = rz^{r-1} - 2(r-1)z^{r-2}$.

Example 4.10. *For $r = 3$, $d = h = 1$ the characteristic polynomial $R(z) = z^3 - 2z^2 - 1$ associated to $\mathfrak{R}_3 = \{\{R_n^{(s)}\}_{n\geq 0},\ 1 \leq s \leq 3\}$ has roots $\lambda_1 \approx 2.2056$, $\lambda_2 \approx -0.10278 - 0.66546i$ and $\lambda_3 \approx -0.10278 + 0.66546i$. Then , the application of Theorem 4.8 allow us get the follows identities,*

$$R_n = \frac{\lambda_1^{n-1}}{R'(\lambda_1)} + \frac{\lambda_2^{n-1}}{R'(\lambda_2)} + \frac{\lambda_3^{n-1}}{R'(\lambda_3)}, \text{ for } n \geq 1,$$

$$R_n^{(1)} = \frac{\lambda_1^{n-2}}{R'(\lambda_1)} + \frac{\lambda_2^{n-2}}{R'(\lambda_2)} + \frac{\lambda_3^{n-2}}{R'(\lambda_3)}, \text{ for } n \geq 3$$

$$R_n^{(2)} = \frac{\lambda_1^{n-3}}{R'(\lambda_1)} + \frac{\lambda_2^{n-3}}{R'(\lambda_2)} + \frac{\lambda_3^{n-3}}{R'(\lambda_3)}, \text{ for } n \geq s+3,$$

where $R'(z) = 3z^2 - 4z$.

Similar result of Proposition 4.7 can be started for a sequence of generalized Pell numbers of type (3), with arbitrary initial conditions. More precisely, the combination of Proposition 2.8 and Theorem 4.8 permit us to obtain the analytical expression for the general setting of a sequence generalized Pell numbers $\{W_n\}_{n\geq 0}$ of type (3), with arbitrary initial conditions α_0, α_1, ..., α_{r-1}, without solving the related Vandermonde linear system. That is, we have $W_n = \sum_{k=1}^{r} \alpha_k R_n^{(k)}$, where the analytical expressions of the $R_n^{(k)}$, with $R_n^{(r)} = R_n$, are given as in Theorem 4.8.

4.2. **Study of the case $d = 0$: Generalized Fibonacci numbers.** As mentioned above in Remark 2.4, when $d = 0$ then Equation (2) is nothing else but that defining the $r-$generalized Fibonacci numbers $\{F_n\}_{n\geq 0}$ studied in [15]. And $P(z) = z^r - z^{r-1} - z^{r-2} - ... - z - 1$ is the characteristic polynomial associated to $r-$generalized Fibonacci numbers. The following results show us that $P(z)$ has simple roots.

Lemma 4.11. *The roots of the polynomial*

$$P(z) = z^r - z^{r-1} - z^{r-2} - ... - z - 1,$$

are simple.

Proof. The polynomial $P(z) = z^r - z^{r-1} - z^{r-2} - ... - z - 1$ can be written under the form $P(z) = z^r - (z^{r-1} + ... + z + 1) = z^r - \frac{z^r - 1}{z-1}$. Since $P(1) \neq 0$, we have

$$P(z) = \frac{z^{r-1} - 2z^r + 1}{z - 1} = \frac{S(z)}{z - 1},$$

where $S(z) = z^{r+1} - 2z^r + 1$. Since $P(1) \neq 0$, we show easily that $P(\lambda) = 0$ if, and only if, $S(\lambda) = 0$, or equivalently,

$$\lambda^{r+1} - 2\lambda^r + 1 = 0. \tag{25}$$

Suppose that λ is a root of $P(z)$, with multiplicity $m \geq 2$, thus $P'(\lambda) = 0$. Since $S(\lambda) = 0$, $\lambda \neq 1$ and $P'(z) = \frac{S(z) - S'(z)(z-1)}{(z-1)^2}$, where $P'(z)$ denote the derivative of $P(z)$, we derive $S'(\lambda) = 0$, namely, we have the

$$S'(\lambda) = [(r + 1)\lambda - 2r]\lambda^{r-1} = 0.$$

Since $P(0) = -1 \neq 0$ then $\lambda \neq 0$. Hence, we derive the following the equality,

$$\lambda = \frac{2r}{r + 1}, \tag{26}$$

By Expression (25) we obtain $\lambda^{r+1} - 2\lambda^r + 1 = (\lambda - 2)\lambda^r + 1 = 0$. Therefore, we have $\left(\frac{2r}{r+1} - 2\right)\lambda^r = \left(\frac{-2}{r+1}\right)\lambda^r = -1$. Following Expression (26), we get $\lambda^{r+1} = \left(\frac{2r}{r+1}\right)^{r+1} = r$. For $r = 2$, we have $\lambda^3 = \frac{64}{27} > 2$. Suppose that $\lambda^{r+1} > r$. Since $r \geq 2$ we have $\lambda = \frac{2r}{r+1} = 2 - \frac{1}{r+1} > 1,5$. Therefore, we have,

$$\lambda^{r+2} = \lambda^{r+1} \times \lambda > 1,5r = r + 0,5r \geq r + 1.$$

Therefore, the equality $\lambda^{r+1} = \left(\frac{2r}{r+1}\right)^{r+1} = r$ is not valid. Consequently, the roots of the polynomial $P(z) = z^r - z^{r-1} - z^{r-2} - ... - z - 1$ are simple.

□

Therefore, results of Section 4.1 are still valid for the generalized Fibonacci numbers and generalized Fibonacci r-numbers. More precisely, by considering the parameter $d = 0$, we have the analytical formula to each element of Fibonacci fundamental system $\{\{F_n^{(s)}\}_{n\geq 0},\ 1 \leq s \leq r\}$ described in [Section 2, [15]], namely, we have the following results.

Theorem 4.12. *Let $\{\{F_n^{(s)}\}_{n\geq 0},\ 1 \leq s \leq r\}$ be the Fibonacci fundamental system. Then, the analytical expression of each $F_n^{(s)} (1 \leq s \leq r)$ is given by,*

$$F_n = \sum_{i=1}^{r} \frac{\lambda_i^n}{P'(\lambda_i)} = \sum_{i=1}^{r} \frac{\lambda_i^n}{\prod\limits_{k\neq i}(\lambda_i - \lambda_k)}, \textit{for } n \geq r,$$

$$F_n^{(1)} = \sum_{i=1}^{r} \frac{\lambda_i^{n-1}}{P'(\lambda_i)} = \sum_{i=1}^{r} \frac{\lambda_i^{n-1}}{\prod\limits_{k\neq i}(\lambda_i - \lambda_k)}, \textit{for } n \geq r + 1,$$

$$F_n^{(s)} = \sum_{j=1}^{s}\sum_{i=1}^{r} \frac{\lambda_i^{n+s-j-1}}{P'(\lambda_i)} = \sum_{j=1}^{s}\sum_{i=1}^{r} \frac{\lambda_i^{n+s-j-1}}{\prod\limits_{k\neq i}(\lambda_i - \lambda_k)}, \textit{for } r \geq r + s,$$

where $\lambda_1, \cdots, \lambda_r$ the simple roots of the polynomial $P(z) = z^r - z^{r-1} - z^{r-2} - ... - z - 1$.

The analytical formulas describe in Theorem 4.12 was not established in [15]. In best of our knowledge the analytical representations presented in Theorem 4.12 are new in the literature.

Observe that the Lemma 4.3 is still valid for parameter $d = 0$, its proof is derived from a direct computation. Then, Proposition 4.8 still also for this parameter. Specially, we can establish the analytical formulas for the fundamental system $\{\{R_n^{(s)}\}_{n\geq 0},\ 1 \leq s \leq r\}$ related to the sequences of Fibonacci r-numbers, with parameters $h = 1$ and $d = 0$, namely,

$$R_n = \sum_{i=1}^{r} \frac{\lambda_i^{n-1}}{R'(\lambda_i)},\ R_n^{(1)} = \sum_{i=1}^{r} \frac{\lambda_i^{n-2}}{R'(\lambda_i)},\ \text{for } n \geq r, \text{and } R_n^{(s)} = \sum_{i=1}^{r} \frac{\lambda_i^{n-s-1}}{R'(\lambda_i)},\ \text{for } n \geq r+s,$$

where $\lambda_1, \cdots, \lambda_r$ the simple roots of the polynomial $R(z) = z^r - z^{r-1} - 1$.

Taking into account the preceding data, we can assert also that Propositions 4.7 and 4.9 are still valid for r-generalized Fibonacci numbers and generalized Fibonacci r-numbers.

5. Analytic representation of Pell numbers (1): Some special cases with $1 \leq i \leq r-3$

For the linear recursive equation defining the model of generalized Pell numbers (1), the characteristic polynomial is given as follows,

$$P(z) = z^r - 2^d z^{r-1} - z^{r-i-2} - \ldots - z - h. \tag{27}$$

We study here the analytic expression of some special cases of the the model of generalized Pell numbers (1), by establishing that the roots of their associated characteristic polynomial are simple. To reach our goal, we consider the notion of Sylvester matrix.

It is well-known that a Sylvester matrix is a matrix associated with two univariate polynomials $P(z)$ and $Q(z)$, whose entries are given by coefficients of these two polynomials [9]. When the determinant of the Sylvester matrix $S_{P,Q}$, called the resultant, is zero, then the two polynomials have a common root (in case of coefficients in a field) or a non-constant common divisor (in case of coefficients in an integral domain). Considering the polynomial (27) and its derivative $P'(z)$, if the determinant of the Sylvester matrix $S_{P,P'}$ is different from 0, then the polynomials $P(z)$ and $P'(z)$ don't have common roots. This means that if $det(S_{P,P'}) \neq 0$, then the roots of $P(z)$ are simple.

5.1. **Special case $r = 4$ and $i = 1$.** Taking $r = 4$, the possible values for i are $0, 1$ and 2. The cases $i = 0$ and $i = 2$, with $h = 1$, have been studied in the previous section as basic cases. Let consider $r = 4, i = 1$ and positive integers d and h in Expression (27). The associated characteristic polynomial is $P(z) = z^4 - 2^d z^3 - z - h$, with derivative $P'(z) = 4z^3 - 32^d z^2 - 1$. In this special case the Sylvester matrix associated to P and P' is given by,

$$S_{P,P'} = \begin{pmatrix} 1 & -2^d & 0 & -1 & -h & 0 & 0 \\ 0 & 1 & -2^d & 0 & -1 & -h & 0 \\ 0 & 0 & 1 & -2^d & 0 & -1 & -h \\ 4 & -3*2^d & 0 & -1 & 0 & 0 & 0 \\ 0 & 4 & -3*2^d & 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & -3*2^d & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 & -3*2^d & 0 & -1 \end{pmatrix}.$$

Computational calculus [1] give us that, for all h, d positive integers, the determinant of $S_{P,P'}$ is equal the $g(d,h) = -27(2^{3d})(2^d)h^2 - 192(2^d)(h^2) - 256(h^3) - 22(2^{2d})(2^d) - 6(2^{2d})h + 18(2^{3d}) - 27$.

Lemma 5.1. *There is no positive integers solutions to the equation* $-27(2^{3d})(2^d)h^2 - 192(2^d)(h^2) - 256(h^3) - 22(2^{2d})(2^d) - 6(2^{2d})h + 18(2^{3d}) - 27 = 0$.

Proof. In fact, taking $x = 2^d$, the equation $g(d,h) = 0$ is equivalent to $-256h^3 - (27x^4 + 192x)h^2 - 6x^2h - 4x^3 - 27 = 0$ that have as real solutions the pairs (x,h) given by $(1,1)$, $\left(0, -\frac{4}{2^{2/3}}\right)$. The last possibility are not applied because x is a function with image in positive reals. Then, the only integer solution is $(1,1)$, but in this case, $x = 1 = 2^d$ or $d = 0$, what is impossible since d is a positive integer. [2] □

Lemma 5.1 show us that the determinant of $S_{P,P'}$ is different from zero for all positive integers d and h. Then, follows the result.

Lemma 5.2. *The roots of the polynomial $P(z) = z^4 - 2^d z^3 - z - h$, are simple, for d and h positive integers.*

Therefore, for $r = 4$ and $i = 1$ we can established the analogous statements given in Section 4. Indeed, using Proposition 2.5 and Lemmas 5.1- 5.2, we can formulate the following result.

Theorem 5.3. *Let $\{\{P_n^{(s)}\}_{n\geq 0},\ 1 \leq s \leq 4\}$ be the fundamental system defined as in* (10). *Then, the analytic expression of each $P_n^{(s)}, (1 \leq s \leq 4)$ is given by,*

$$P_{1,n} = \rho_1(n+1,4) = \sum_{j=1}^{4} \frac{\lambda_j^{n-1}}{P'(\lambda_j)} = \sum_{j=1}^{4} \frac{\lambda_j^{n-1}}{\prod_{k\neq j}(\lambda_j - \lambda_k)},\ \textit{for } n \geq 4,$$

$$P_{1,n}^{(1)} = hP_{n-1} = h\rho_1(n,4) = h\sum_{j=1}^{4} \frac{\lambda_j^{n-2}}{P'(\lambda_j)} = h\sum_{j=1}^{4} \frac{\lambda_j^{n-2}}{\prod_{k\neq j}(\lambda_j - \lambda_k)},\ \textit{for } n \geq 5,$$

and for $n \geq 4+s$ and $2 \leq s \leq 3$, we have,

$$P_{1,n}^{(s)} = \sum_{j=s-2}^{s-2} \rho_1(n-j,4) + h\rho_1(n-s+1,4) = \sum_{t=1}^{4} \frac{\lambda_t^{n-s}}{P'(\lambda_t)} + h\frac{\lambda_t^{n-s-1}}{P'(\lambda_t)},$$

where $\lambda_1, \cdots, \lambda_4$ are the simple roots of the polynomial $P(z) = z^4 - 2^d z^3 - z - h$.

[1] the algebraic results was obtained using software Matlab and CoCalc-SageMath

[2] the numeric results was obtained using software Matlab and CoCalc-SageMath

We have the following illustrative numerical example of Theorem 5.3.

Example 5.4. *For parameters $h = 2$ and $d = 5$ the resultant (namely, the determinant of the associated Sylvester matrix) is equal $-113391643 \neq 0$, then the roots of $P(z) = z^4 - 32z^3 - z - 2$ are simple and we have $\lambda_1 \approx -0,36933$, $\lambda_2 \approx 32,001$, $\lambda_3 \approx 0,18414 - 0,36785i$ and $\lambda_4 \approx 0,18414 + 0,36785i$. Therefore, the analytic expression of each $P_n^{(s)} (1 \leq s \leq 4)$ is given by,*

$$P_{1,n} = \rho_1(n+1,4) = \sum_{j=1}^{4} \frac{\lambda_j^{n-1}}{P'(\lambda_j)} = \sum_{j=1}^{4} \frac{\lambda_j^{n-1}}{\prod_{k\neq j} (\lambda_j - \lambda_k)}, \textit{ for } n \geq 4,$$

$$P_{1,n}^{(1)} = 2P_{n-1} = 2\rho_1(n,4) = 2\sum_{j=1}^{4} \frac{\lambda_j^{n-2}}{P'(\lambda_j)} = 2\sum_{j=1}^{4} \frac{\lambda_j^{n-2}}{\prod_{k\neq j} (\lambda_j - \lambda_k)}, \textit{ for } n \geq 5,$$

$$P_{1,n}^{(2)} = \rho_1(n,4) + 2\rho_1(n-1,4) = \sum_{t=1}^{4} \frac{\lambda_t^{n-2}}{P'(\lambda_t)} + 2\frac{\lambda_t^{n-3}}{P'(\lambda_t)}, \textit{ for } n \geq 6,$$

$$P_{1,n}^{(3)} = \sum_{j=1}^{1} \rho_1(n-j,4) + h\rho_1(n-2,4) = \sum_{t=1}^{4} \frac{\lambda_t^{n-3}}{P'(\lambda_t)} + 2\frac{\lambda_t^{n-4}}{P'(\lambda_t)}, \textit{ for } n \geq 7.$$

For $r = 4$ in the model of generalized Pell numbers (1), a similar result of Proposition 4.7 can be started for a sequence of generalized Pell numbers with arbitrary initial conditions. More precisely, the combination of Proposition 2.8 and Theorem 5.3 permit us to get the analytical expression for the general setting of a sequence generalized Pell numbers $\{W_n\}_{n\geq 0}$, with arbitrary initial conditions α_0, α_1,α_2, α_3, without solve the Vandermonde determinant system. That is, we have $W_n = \sum_{k=1}^{4} \alpha_k P_n^{(k)}$, where the analytical expressions of the $P_n^{(k)}$ $(1 \leq k \leq 4)$, with $P_n^{(4)} = P_n$, are given as in Theorem 5.3.

Note that, Theorem 5.3 is about a result established on a special case of the model of the generalized Pell numbers (1), which is not current in the literature.

5.2. **Special case** $r = 5$ **and** $i = 1$. Let consider $r = 5, i = 1$ and positive integers d and h. The associated characteristic polynomial is $P(z) = z^5 - 2^d z^4 - z^2 - z - h$, with derivative $P'(z) = 5z^4 - 4(2^d)z^3 - 2z - 1$. The Sylvester matrix associated to $P(z)$ and $P'(z)$ is,

$$S_{P,P'} = \begin{pmatrix} 1 & -2^d & 0 & -1 & -1 & -h & 0 & 0 & 0 \\ 0 & 1 & -2^d & 0 & -1 & -1 & -h & 0 & 0 \\ 0 & 0 & 1 & -2^d & 0 & -1 & -1 & -h & 0 \\ 0 & 0 & 0 & 1 & -2^d & 0 & -1 & -1 & -h \\ 5 & -4*2^d & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 5 & -4*2^d & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 5 & -4*2^d & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & -4*2^d & 0 & -1 & -1 \end{pmatrix}$$

The determinant of $S_{P,P'}$, namely, $g(h,d) = det(S_{P,P'})$ is equal,

$$\begin{aligned} g(h,d) &= -256(2^{3d})(2^{2d})h^3 - 480(2^{2d})(2^d)h^2 - 2000(2^{2d})h^3 - 2500(2^d)h^3 \\ &+ 3125h^4 - 112(2^{3d})(2^d)h - 292(2^{2d}(2^d)h - 128(2^{4d})h^2 + 320(2^{3d})h^2 \\ &- 50(2^{2d})h^2 - 900(2^d)h^2 - 32(2^{3d})2^d + 60(2^{2d})2^d - 32(2^{4d})h \\ &+ 240(2^{3d})h + 24(2^{2d})h - 1020(2^d)h - 2250h^2 + 5(2^{4d}) \\ &- 64(2^{3d}) + 6(2^{2d}) - 192(2^d) - 1708h - 283. \end{aligned}$$

Taking $x = 2^d$, the equation $g(d,h) = 0$ is equivalent to

$$\begin{aligned} &- 256x^5h^3 - 480x^3h^2 - 2000x^2h^3 - 2500xh^3 + 3125h^4 - 112x^4h - 292x^3h - 128x^4h^2 \\ &+ 320x^3h^2 - 50x^2h^2 - 900xh^2 - 32x^4 + 60x^3 - 32x^4h + 240x^3h + 24x^2h - 1020xh \\ &- 2250h^2 + 5x^4 - 64x^3 + 6x^2 - 192x - 1708h - 283 = 0. \end{aligned}$$

The direct computational verification give us the following lemma.

Lemma 5.5. *There is no positive integers solutions of the equation* $-256x^5h^3 - 480x^3h^2 - 2000x^2h^3 - 2500xh^3 + 3125h^4 - 112x^4h - 292x^3h - 128x^4h^2 + 320x^3h^2 - 50x^2h^2 - 900xh^2 - 32x^4 + 60x^3 - 32x^4h + 240x^3h + 24x^2h - 1020xh - 2250h^2 + 5x^4 - 64x^3 + 6x^2 - 192x - 1708h - 283 = 0.$

Then, Proposition 5.5 shows us that the determinant of $S_{P,P'}$ is different from zero, for any positive integers d and h. Therefore, we have the following lemma.

Lemma 5.6. *The roots of the polynomial $P(z) = z^5 - 2^dz^4 - z^2 - z - h$ are simple, for every positive integers d and h.*

As a consequence of the two Lemmas 5.5 and 5.6, the analogous statements given in Section 4, can be formulated as follows.

Theorem 5.7. *Let $\{\{P_n^{(s)}\}_{n\geq 0},\ 1 \leq s \leq 5\}$ be the Pell fundamental system defined as in* (10). *Then, the analytic expression of each $\{P_n^{(s)}\}_{n\geq 0},\ 1 \leq s \leq 5\}$ is given by,*

$$P_n = \rho_1(n+1,5) = \sum_{j=1}^{5} \frac{\lambda_j^{n-1}}{P'(\lambda_j)} = \sum_{j=1}^{5} \frac{\lambda_j^{n-1}}{\prod\limits_{k\neq j}(\lambda_j - \lambda_k)},\ \textit{for } n \geq 5,$$

$$P_n^{(1)} = hP_{n-1} = h\rho_1(n,5) = h\sum_{j=1}^{5} \frac{\lambda_j^{n-2}}{P'(\lambda_j)} = h\sum_{j=1}^{4} \frac{\lambda_j^{n-2}}{\prod\limits_{k\neq j}(\lambda_j - \lambda_k)},\ \textit{for } n \geq 6,$$

and for $n \geq 5 + s$ $(2 \leq s \leq 4)$, we have,

$$P_n^{(s)} = \sum_{j=s-3}^{s-2} \rho_1(n-j,5) + h\rho_1(n-s+1,5) = \sum_{j=s-3}^{s-2} \sum_{t=1}^{5} \frac{\lambda_t^{n-j-2}}{P'(\lambda_t)} + h\frac{\lambda_t^{n-s-1}}{P'(\lambda_t)},$$

where $\lambda_1, \cdots, \lambda_5$ are the simple roots of the polynomial $P(z) = z^5 - 2^dz^4 - z^2 - z - h$.

The following numerical example is for illustrating the content of Theorem 5.7.

Example 5.8. *For the parameters $h = 3$ and $d = 5$, the resultant is equal $231159740436 \neq 0$. Hence, the roots of the polynomial $P(z) = z^5 - 32z^4 - z - 3$ are simple, that is, we have $\lambda_1 \approx 32.0000$, $\lambda_2 \approx -0.391434 - 0.363206i$, $\lambda_3 \approx -0.391434 + 0.363206i$, $\lambda_4 \approx 0.391418 - 0.419021i$, and $\lambda_5 \approx 0.391418 + 0.419021i$. Then, the analytic expression of each $P_n^{(s)} (1 \leq s \leq 5)$ is given by,*

$$P_n = \rho_1(n+1,5) = \sum_{j=1}^{5} \frac{\lambda_j^{n-1}}{P'(\lambda_j)} = \sum_{j=1}^{5} \frac{\lambda_j^{n-1}}{\prod_{k\neq j}(\lambda_j - \lambda_k)}, \text{ for } n \geq 5,$$

$$P_n^{(1)} = 3P_{n-1} = 2\rho_1(n,5) = 3\sum_{j=1}^{5} \frac{\lambda_j^{n-2}}{P'(\lambda_j)} = 3\sum_{j=1}^{5} \frac{\lambda_j^{n-2}}{\prod_{k\neq j}(\lambda_j - \lambda_k)}, \text{ for } n \geq 6,$$

and for $n \geq 5 + s$ and $2 \leq s \leq 4$, we have,

$$P_n^{(2)} = \rho_1(n,5) + 3\rho_1(n-1,5) = \sum_{t=1}^{5} \frac{\lambda_t^{n-2}}{P'(\lambda_t)} + 3\frac{\lambda_t^{n-3}}{P'(\lambda_t)},$$

$$P_n^{(3)} = \sum_{j=0}^{1} \rho_1(n-j,5) + 3\rho_1(n-s+1,5) = \sum_{j=0}^{1}\sum_{t=1}^{5} \frac{\lambda_t^{n-j-2}}{P'(\lambda_t)} + 3\frac{\lambda_t^{n-4}}{P'(\lambda_t)},$$

$$P_n^{(4)} = \sum_{j=1}^{2} \rho_1(n-j,5) + 3\rho_1(n-s+1,5) = \sum_{j=1}^{2}\sum_{t=1}^{5} \frac{\lambda_t^{n-j-2}}{P'(\lambda_t)} + 3\frac{\lambda_t^{n-5}}{P'(\lambda_t)}.$$

5.3. **Special case $r = 5$ and $i = 2$.** Suppose that $r = 5, i = 2$ and consider the positive integers h and d. The associated characteristic polynomial is $P(z) = z^5 - 2^d z^4 - z - h$, with derivative $P'(z) = 5z^4 - 4(2^d)z^3 - 1$. Then, the determinant of the associated Sylvester matrix is given by,

$$\begin{aligned} g(h,d) = {} & 256(2^{4x})(2^x)h^3 + 2500(2^x)h^3 + 3125h^4 + 436(2^{2x})(2^x)h - \\ & 50(2^{2x}h)^2 - 336(2^{3x})2^x - 400(2^{3x})h + 309(2^{4x}) - 256. \end{aligned}$$

The same variable change $x = 2^d$, gives us the equation

$$256(h^3)u^5 + 3125h^4 + 2500(h^3)u - 50(h^2)u^2 + 36hu^3 - 27u^4 - 256 = 0.$$

We can verify that, the only integers solutions the preceding equation are given by $h = -1$, $u = 1$, and $h = 1$, $u = -1$,, which is impossible. Then, it follows the result showing us that the resultant, namely, determinant of Sylvester matrix $S_{P,P'}$ is different from zero for every positive integers d and h.

Lemma 5.9. *There is no positive integers solutions to the equation $-256(2^{4d})(2^d)h^3 - 2500(2^d)h^3 + 3125h^4 - 436(2^{2d})(2^d)h - 502^{2d}h^2 - 336(2^{3d})2^d + 400(2^{3d})h + 3092^{4d} - 256 = 0$.*

As a consequence of the Lemma 5.9, we deduce the following proposition.

Proposition 5.10. *The roots of the polynomial*

$$P(z) = z^5 - 2^d z^4 - z - h$$

are simple, for d and h positive integers.

Therefore, with the aid of Lemma 5.9 an Proposition 5.10, the following theorem is valid.

Theorem 5.11. *Let* $\{\{P_n^{(s)}\}_{n\geq 0},\ 1\leq s\leq 5\}$ *be the Pell fundamental system defined as in* (10). *Then, the analytic expression of each* $P_n^{(s)}(1\leq s\leq 5)$ *is given by,*

$$P_{2,n}=\rho_2(n+1,5)=\sum_{j=1}^{5}\frac{\lambda_j^{n-1}}{P'(\lambda_j)}=\sum_{j=1}^{5}\frac{\lambda_j^{n-1}}{\prod\limits_{k\neq j}(\lambda_j-\lambda_k)},\ \textit{for } n\geq 5,$$

$$P_{2,n}^{(1)}=hP_{n-1}=h\rho_2(n,5)=h\sum_{j=1}^{5}\frac{\lambda_j^{n-2}}{P'(\lambda_j)}=h\sum_{j=1}^{4}\frac{\lambda_j^{n-2}}{\prod\limits_{k\neq j}(\lambda_j-\lambda_k)},\ \textit{for } n\geq 6,$$

and for $n\geq 5+s$ $(2\leq s\leq 4)$, *we have,*

$$P_{2,n}^{(s)}=\sum_{j=s-2}^{s-2}\rho_2(n-j,5)+h\rho_2(n-s+1,5)=\sum_{t=1}^{5}\frac{\lambda_t^{n-s}}{P'(\lambda_t)}+h\frac{\lambda_t^{n-s-1}}{P'(\lambda_t)},$$

where $\lambda_1,\cdots,\lambda_5$ *are the simple roots of the polynomial* $P(z)=z^5-2^dz^4-z-h$.

As an illustrative application of Theorem 5.11, we give the following numerical example.

Example 5.12. *For the parameters* $h=2$ *and* $d=6$, *the resultant is equal* $62810960\neq 0$. *Then, the roots of the associated polynomial* $P(z)=z^5-64z^4-z-2$ *are simple, that is, we have* $\lambda_1\approx 64.0000$, $\lambda_2\approx -0.297632-0.274181i$, $\lambda_3\approx -0.297632+0.274181i$, $\lambda_4\approx 0.297630-0.319757i$, *and* $\lambda_5\approx 0.297630+0.319757i$. *Then, the analytic expression of each* $P_n^{(s)}(1\leq s\leq 5)$ *is given by,*

$$P_{2,n}=\rho_2(n+1,5)=\sum_{j=1}^{5}\frac{\lambda_j^{n-1}}{P'(\lambda_j)}=\sum_{j=1}^{5}\frac{\lambda_j^{n-1}}{\prod\limits_{k\neq j}(\lambda_j-\lambda_k)},\textit{for } n\geq 5,$$

$$P_{2,n}^{(1)}=3P_{n-1}=3\rho_2(n,5)=3\sum_{j=1}^{5}\frac{\lambda_j^{n-2}}{P'(\lambda_j)}=3\sum_{j=1}^{5}\frac{\lambda_j^{n-2}}{\prod\limits_{k\neq j}(\lambda_j-\lambda_k)},\textit{for } n\geq 6,$$

and for $n\geq 5+s$ *and* $2\leq s\leq 4$, *we have,*

$$P_{2,n}^{(2)}=\rho_2(n,5)+3\rho_2(n-1,5)=\sum_{t=1}^{5}\frac{\lambda_t^{n-2}}{P'(\lambda_t)}+3\frac{\lambda_t^{n-3}}{P'(\lambda_t)},$$

$$P_{2,n}^{(3)}=\sum_{j=1}^{1}\rho_2(n-j,5)+3\rho_2(n-2,5)=\sum_{t=1}^{5}\frac{\lambda_t^{n-3}}{P'(\lambda_t)}+3\frac{\lambda_t^{n-4}}{P'(\lambda_t)},$$

$$P_{2,n}^{(4)}=\sum_{j=2}^{2}\rho_2(n-j,5)+3\rho_2(n-3,5)=\sum_{t=1}^{5}\frac{\lambda_t^{n-4}}{P'(\lambda_t)}+3\frac{\lambda_t^{n-5}}{P'(\lambda_t)}.$$

Remark 5.13. *It is important to note that for Lemmas 5.2, 5.6 and Proposition 5.10, we succeeded another alternative proof, based on the same process considered for establishing Lemmas 4.1 and 4.3.*

It seems for us that, the results of Theorems 5.7 and 5.11 concerning the special case $r = 5$ and $i = 1$ or $i = 2$, of the model of generalized Pell numbers (1), are not current in the literature. Moreover, for $r = 5$ in the model of generalized Pell numbers (1), a similar result of Proposition 4.7 can be formulated for sequences of generalized Pell numbers with arbitrary initial conditions. More precisely, for $r = 5$ the combination of Proposition 2.8 and Theorems 5.7 and 5.11 permit us to get the analytical expression for the general setting of sequence generalized Pell numbers.

6. Concluding remarks and Perspectives

In this paper we have studied the model of generalized Pell numbers (1), where some combinatorial representations of the generalized Pell numbers (1) are provided. Moreover, some identities and combinatorial identities for the model of generalized Pell numbers are established. On the other hand, analytic formulas of a large class of sequences of the model of generalized Pell numbers (1), namely, (2)-(3), are established, without using the usual method of determinant. And in the context of some special cases the use of the determinant of Sylvester matrix allow us to obtain new results. It seems for us that the study of the analytical aspect of the generalized Pell numbers (2)-(3) and for the general model (1), the study of the two special cases $r = 4$, $i = 1$ and $r = 5$, $i = 1$ or $i = 2$, represent an interesting contribution to this model. Several results of our study are not current in the literature.

The analytic approach for the model of generalized Pell numbers (1) can be deepened in order to generalize the results of Section 5. That is, the same process and study with computational effort can be done for all r and i, for $1 \leq i \leq r - 3$.

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