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# Comparative study of computational methods of structural reliability assessment

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***Abstract* — *Safety is an essential requirement of a structural system. Reliability is an additional tool of growing importance in engineering, as it allows us to quantify uncertainties in the design. Thus, reliability assists us in making more suitable decisions regarding the safety of a structure. The present work compares and analyzes structural reliability methods applied to various examples of limit state functions. These methods are essential tools for this analysis because they identify and quantify uncertainties in random variables, allowing the evaluation of the probability of failure of the structure. Structural reliability methods were programmed and simulated in the Python language. The performance of these methods was analyzed through examples of linear, nonlinear, implicit, and explicit limit state functions. The results indicate that the simulation of Monte de Carlo brute force (MCBF) and importance sampling (MCAI) proved to be quite efficient for the examples studied in this work, with values equal to or very close to the reference values from the literature. The First Order and Second Moment Method (FOSM) presented limitations in some examples when the basic random variables do not have a normal distribution and the limit state function is nonlinear. The first-order reliability method (FORM) employs a failure surface linearization, which does not work well for highly nonlinear problems. The second-order reliability method (SORM) has improved the FORM results by including additional information about the curvature of the limit state function.*

## I. INTRODUCTION

According to Haldar and Mahadevan [1], most observable phenomena contain a certain amount of uncertainty. In general, repeated measurements of physical phenomena yield multiple results. The occurrence of various outcomes with no pattern is described by terms such as uncertainty and randomness.

According to Real *et al.* [2], the main quantities involved in engineering design are, in reality, random variables, which have a probability distribution. Therefore, the structure's response to loading will also be a random variable.

Thus, ensuring adequate safety levels in structures, and providing good performance, is one of the most significant engineering challenges. Models based on limit states are used in the design standards, where a series of safety criteria is specified in the calculation process of structures. The effect of uncertainties on the safety of structures can be diminished by analyzing the probability of failure of a limit state. (ALBUQUERQUE AND REAL [3]).

For Verma *et al*. [4], reliability is defined as the probability of trouble-free performance under established conditions. Still, it can be considered the probability that an element can perform its intended function for a specified time under pre-established design conditions.

After Beck [5], due to the uncertainties of the parameters of engineering projects, it is necessary to carry out reliability analyzes because of possible failures. The answer, in this case, is the determination of the failure probability or reliability index associated with a limit state function that depends on the random variables of the problem. That way, structural reliability methods are essential tools for this analysis, as they identify and quantify uncertainties in random variables.

According to Santos [6], several methods have been developed to estimate the failure probability and reliability index values. Among these methods, FOSM (First Order Second Moment), FORM (First Order Reliability Method), and SORM (Second Order Reliability Method) stand out. These structural reliability methods, also known as transformation methods, approximate the limit state function at the design point by a hyperplane, in the case of FOSM and FORM. In the case of SORM, a quadratic hypersurface is used. The approximation of these methods can lead to errors, especially when the limit state function is strongly nonlinear. In this way, Monte Carlo simulation can estimate the probability of failure and the correct reliability index, as this method is considered the most exact.

The Brazilian Association of Technical Standards (ABNT) is the main standardization body in the country. The Brazilian Norms (NBR) are technical documents used to standardize and guarantee a quality level. There are still no technical specifications based on structural reliability principles in Brazil, although this is a reality in several other countries. Therefore, this work is justified by the omission of the NBR concerning structural reliability. This paper aims to compare structural reliability methods applied to examples of limit state functions and, thus, analyze the efficiency of the method application.

## II. MATERIAL AND METHOD

The methodology used in the elaboration of this work consists of the development of codes for structural reliability analysis through the methods Monte Carlo Brute Force (MCBF), Monte Carlo with Importance Sampling (MCIS), FOSM, FORM with the HLRF algorithms (Hasofer, Lind, Rackwitz and Fiessler) and iHLRF (improved Hasofer, Lind, Rackwitz and Fiessler) and SORM.

After programming, a comparative study between the methods was presented, where eighteen examples of linear, nonlinear, and implicit limit state functions obtained by some researchers are considered. Then, the performance and shortcomings of each method are analyzed.

### 2.1 – Limit State

Beck [5] states that each distinct form of arriving at an undesirable state is generically called a failure mode. Where each failure mode gives rise to a limit state equation $g(X)$, which is a function of the vector of random variables $X = \{X_1, X_2, \dots, X_n\}$, in such a way that the limit state is determined by equation 1:

$$g(X) = g(X_1, X_2, \dots, X_n) = 0 \quad (1)$$

a limit state equation $g(X)$ is written in such a way as to divide the problem domain into failure $\Omega_f$ and safety domains $\Omega_s$, as shown in equations 2 and 3:

$$\Omega_f = \{X | g(X) \leq 0\} \quad (2)$$

$$\Omega_s = \{X | g(X) > 0\} \quad (3)$$

The failure domain $\Omega_f$ is constituted of all points in the sample space $X$ that lead to the structure's failure. The secure domain $\Omega_s$ is the complementary set to the failure domain.

### 2.2 – Limit state function

According to Melchers and Beck [8], just as there are uncertainties in engineering projects, a structural system has the risk of not reaching the designed performance. Consequently, the performance of this system cannot be guaranteed to be completely secure. Thus, for Santos [6], it is convenient to define the limit state function; considering a structure with resistance *R* and load *S*, being possible to evaluate the failure probability. A limit state function in a probabilistic analysis can be determined by equation 4:

$$g(x) = R(x) - S(x) \quad (4)$$

Where *x* is the vector of random variables of the problem; $R(x)$ is the random variable representing the strength of the structure; $S(x)$ is the random variable representing the loading.

The probability of failure ($P_f$) is defined when the structure's resistance is less than the loading, and the safety probability ($P_S$) is when the resistance is greater than or equal to the loading, being given by equations 5 and 6:

$$P_f = P[g(x) < 0] \tag{5}$$

$$P_S = 1 - P_f. \tag{6}$$

In compliance with the JCSS [8], the probability of failure must be calculated based on the type of joint distribution of the basic variables and on the standardized distributive formalism of dealing with model uncertainty and statistical uncertainty.

Figure 1 shows a surface with the failure region, the safety region, and the limit state function for a system with two variables.

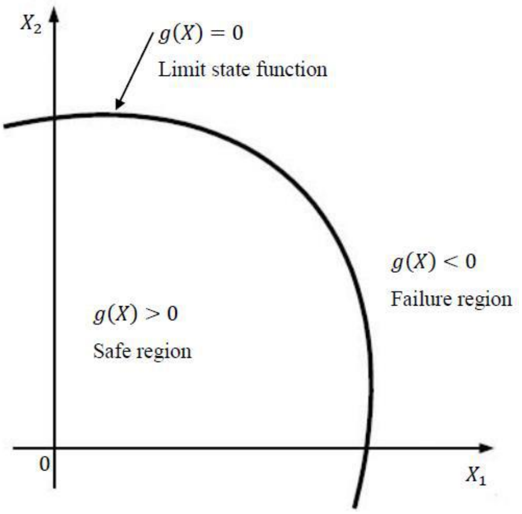


*Fig. 1: Limit state function for a system with two variables (ALBUQUERQUE [7])*

For Beck [5], the reliability index calculated in problems involving statistically independent normally distributed random variables and a linear limit state function corresponds to the smallest distance between the origin and the limit state function and is given through equation 7.

$$\beta = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}. \tag{7}$$

According to Melchers and Beck [8], the failure probability can be obtained directly from the reliability index for linear limit state functions. It can still be defined if the function is nonlinear, but only for an approximate (hyper)tangent plane. In each case, $\beta$ represents the smallest distance between the origin in standardized normal space and the tangent plane, as shown in Figure 2.

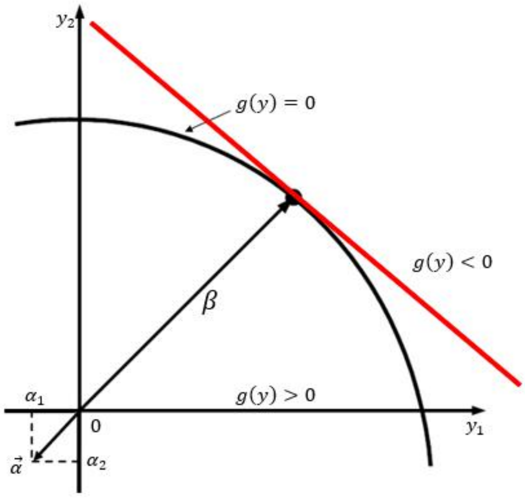


*Fig. 2: Representation of the reliability index (ALBUQUERQUE [7])*

In Figure 2, the minimum distance point on the hyperplane is denominated as the design point. This point has the highest probability density within the space covered by the failure region, according to Melchers and Beck [8].

2.3 – Reliability Methods

Monte Carlo simulation is a numerical method that is based on random sampling. Second Beck [5], in structural engineering, Monte Carlo simulation adapts to linear and nonlinear, static or dynamic problems. In structural reliability, the simulation solves, with the same ease, problems involving a single limit state equation, time-dependent or not.

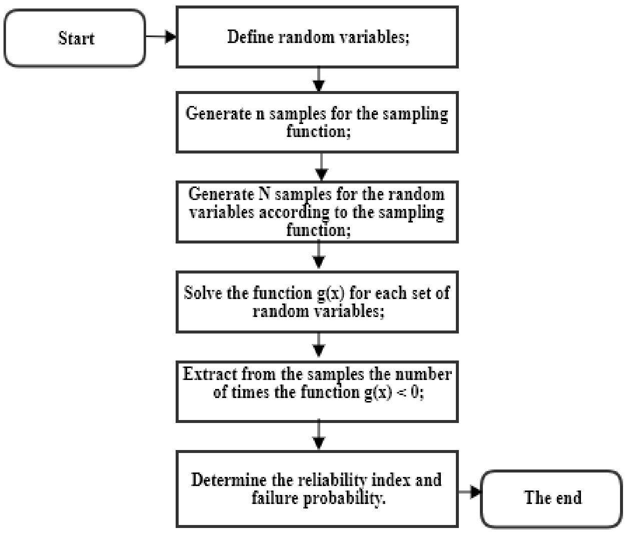


*Fig. 3: Flowchart of the brute force Monte Carlo method algorithm (ALBUQUERQUE [7])*

The brute force Monte Carlo simulation method (MCBF) is a numerical technique based on repetition of the simulation process, using a specific set of values in each simulation.

According to Haldar and Mahadevan [1], the MCBF simulation has six essential elements, and the algorithm of the method is presented in Figure 3.

One of the techniques to accelerate the convergence of Monte Carlo simulation is utilizing importance sampling. The technique determines the probability of failure in the limit state region. Beck [5] states that the importance sampling technique uses a sampling function to avoid generating samples far from the area of interest, far from the failure domain. The points are generated from a sampling function $h_X(x)$. Multiplying and dividing Equation 5 by $h_X(x)$, we get equation 8:

$$P_f \approx \widehat{P_f} = \frac{1}{N}\sum_{i=1}^{N} I_g(x_i)\frac{f_X(x_i)}{h_X(x_i)} \tag{8}$$

The point $x_k$ is associated with a weight $w_k << 1$, represented by the ratio of the functions $f_X(x)$ and $h_X(x)$. The variance of the failure probability is estimated by equation 9:

$$Var[\widehat{P_f}] = \frac{1}{(N-1)}\sum_{k}^{N}(I_g[x_k]w_k - \widehat{P_f})^2 \tag{9}$$

According to Haldar and Mahadevan [1], the algorithm of the MCIS method is presented in Figure 4.

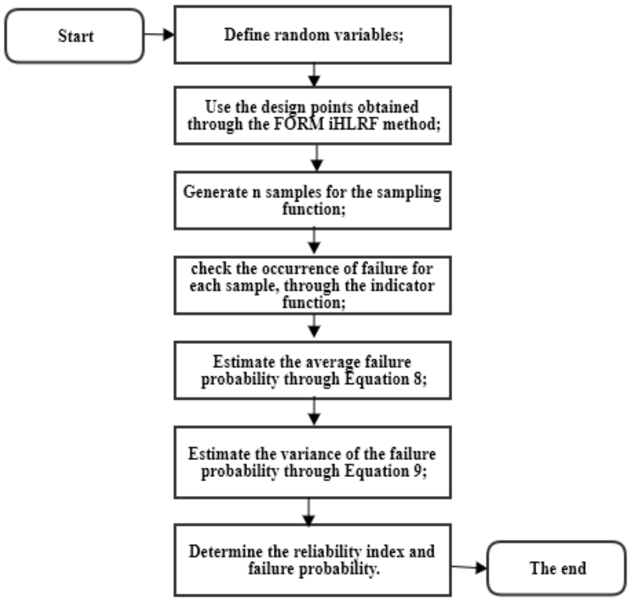


*Fig. 4: Flowchart of the Monte Carlo method algorithm with sampling by importance (ALBUQUERQUE [7])*

According to Haldar and Mahadevan [1], FOSM and FORM can be used to evaluate a function of normal and statistically independent variables. FORM can also represent a nonlinear limit state function by the first-order approximation with equivalent normal variables. And SORM determines the probability of failure by approximating the nonlinear limit state function by a second-order representation. FORM substitutes the limit state function at the design point by a hyperplane, while SORM uses a quadratic hypersurface. In Figure 5, the first and second-order approximations are shown.

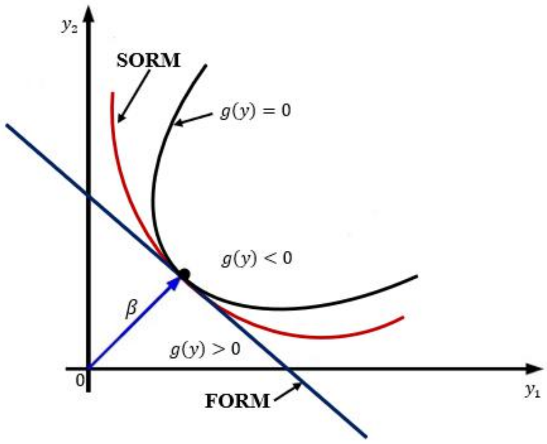


*Fig. 5: First and second-order approximation - FORM e SORM (ALBUQUERQUE [7])*

According to Beck [5], in the FOSM and FORM methods, the limit state equation is approximated by a linear function. The information for constructing the joint probability density function $f_X(x)$ is limited to moments up to second-order (mean and covariance). A representation of the problem variables by their moments corresponds to assuming them as having a normal distribution.

The FOSM method solution algorithm is presented below, following (BECK [5]).

**Algorithm 1.** Reliability method FOSM

1. choice of initial failure point $\mathbf{x}^k$ to k=0 (usually the mean point);
2. evaluation of Jacobian matrices ($J_{yx}$ and $J_{xy}$) and the vector of the means V;
3. point transformation $\mathbf{x}^k$ of $\mathbb{X} \rightarrow \mathbb{Y}$;
4. evaluation of g ($\mathbf{x}^k$);
5. gradient calculation:
   a. calculating the partial derivatives of g(x) in design space $\mathbb{X}$

b. transform the gradient to $\mathbb{Y}$

c. calculating director cosines $\alpha\,(y^k)$

6. calculation of the reliability index $(\beta)$

7. calculation of the new point of failure $y^{k+1}$ through

$$y^{k+1} = \mu_{Xi}^{N} - \alpha_{Xi}\beta\sigma_{X_i}^{N} \tag{10}$$

8. transformation of $y^{k+1}$ in $\mathbb{X}$;

9. verification of the convergence criterion. If:

$$1 - \frac{\nabla g(y^{k+1})\, y^{k+1}}{\|\nabla g(y^{k+1})\|\,\|y^{k+1}\|} < \epsilon \quad e\,|g(y^{k+1})| < \delta \tag{11}$$

being $\epsilon = 10^{-6}$; the algorithm is interrupted ($y^* = y^{k+1}$); otherwise, return to item 4 with *k=k+1* until reaching convergence;

10. at the end, evaluation of the reliability index at the design point: $\beta = \| y^*\|$.

The FORM algorithm is equal to the FOSM for a problem with a normal probability distribution of random variables. When distributions are mixed, it is necessary to calculate the mean and standard deviation of an equivalent normal distribution (ALBUQUERQUE [7]).

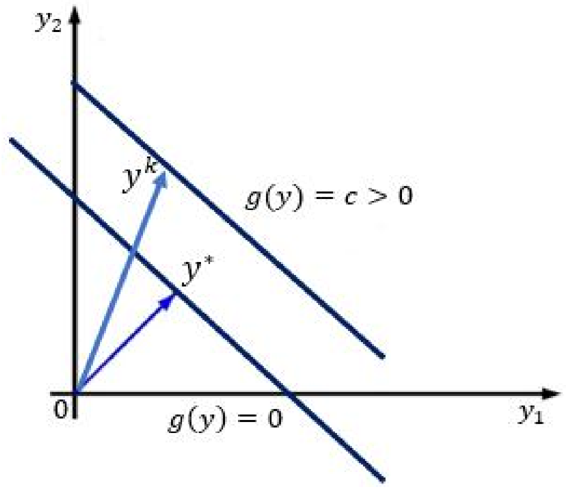


*Fig. 6: Representation of the FORM method with the HLRF algorithm - Linear limit state function (ALBUQUERQUE [7])*

According to Melchers and Beck [8], the search for the design point *y** is one of the fundamental steps to obtaining the failure probability by the FORM method. The transformation of any variables, correlated or not, into statistically independent equivalent normal variables is performed through the Nataf model transformation.

For Beck [5], the solution of reliability problems via FORM for nonlinear limit state functions involves resolving an optimization problem to search for the design point. The most used algorithm is Hasofer, Lind, Rackwitz and Fiessler (HLRF). For nonlinear limit states, the minimum distance calculation becomes an optimization problem. The solution to this problem is presented in equation 12:

$$d = \|y\| = \sqrt{y\, y^t} \tag{12}$$

Figures 6 and 7 illustrate an iteration of the FORM method with the HLRF algorithm, considering a linear and nonlinear limit state function. It is observed that while the design point is not reached, the condition $g(y) = 0$ is not obtained. That is, the point $y^k$ does not reach the failure surface, but a surface to which $g(y)$ is constant (c). Therefore, the optimization algorithm must start from the point $y^k$, which may not be in the limit state, and converge to the minimum distance point $y^*$ in the limit state.

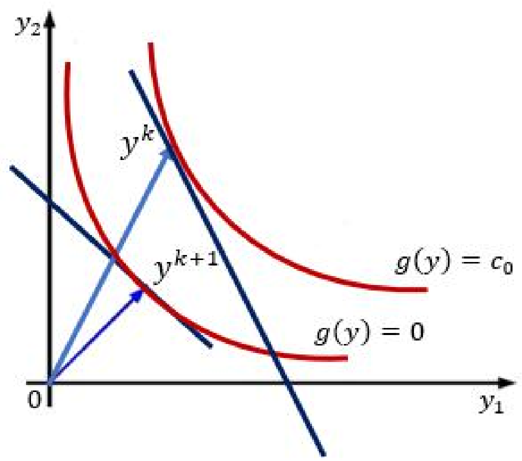


*Fig. 7: Representation of the FORM method with the HLRF algorithm - Nonlinear limit state function (ALBUQUERQUE [7])*

The recurrence formula for $y^{k+1}$ can be obtained considering the linear approximation of $g(y)$, and is represented through equation 13:

$$y^{k+1} = \frac{[\nabla g(y^k)^t y^k - g(y^k)]\nabla g(y^k)}{\|\nabla g(y^k)\|^2}. \quad (13)$$

According to Sudret and Der Kiureghian [10], the HLRF algorithm can be improved (iHLRF) through a linear search for the fit. The original HLRF algorithm gives the search direction of equation 14:

$$d^k = y^{k+1} - y^k = \frac{\nabla g(y^k)^t y^k - g(y^k)}{\|\nabla g(y^k)\|^2} \nabla g(y^k) - y^k \quad (14)$$

The solution algorithm of the FORM iHLRF method is presented following (BECK, [5]).

**Algoritmo 2.** Reliability method FORM iHLRF

Data: $y^k \in \mathbb{R}^n, \Delta > 0, \eta > 1, \alpha \in (0,1], 0 < m_1 < 1$

1. Calculate the search direction of $d^k$ given in equation 14;
2. Determine the penalty parameter c of the merit function

If $|g(y^k)| \geq \Delta$

$$c = \eta \max\left\{\frac{\|y^k\|}{\|\nabla g(y^k)\|}, \frac{1}{2}\frac{\|y^k + d^k\|^2}{|g(y^k)|}\right\}$$

If not

$$c = \eta \frac{\|y^k\|}{\|\nabla g(y^k)\|};$$

3. Determine $\alpha^k$ using inequality

While $p(y^k + \alpha d^k) - p(y^k) > m_1 \alpha \nabla p(y^k)^t d^k$

Do $\alpha = \frac{\alpha}{2}$

$\alpha^k = \alpha;$

4. Update the point y:

The SORM approximates the limit state function at the design point by a quadratic hypersurface. Beck [5] states that this method results in a good approximation of the failure probability when the limit state function, in standard normal space, is linear or weakly nonlinear in the vicinity of the design point.

For Cordeiro [11], the SORM method estimates the failure probability through a second-order approximation of the limit state function at the design point. The second-order derivative of this function approximates the curvature of the limit state function with respect to the equivalent normal space variables. The SORM method algorithm is calculated by obtaining the reliability index and failure probability of the FORM method. The algorithm of the FORM and SORM methods is shown in Figure 8.

According to Haldar and Mahadevan [1], the curvature of any function is related to the second-order derivatives in relation to random variables. Therefore, the second-order reliability method (SORM) improves the FORM result by including additional information about the curvature of the limit state of the function.

## III. RESULTS AND DISCUSSION

Eighteen problems are presented for comparative analysis, applying Monte Carlo simulation and transformation methods in structural reliability problems. The study consists of a comparative analysis of the convergence of the reliability index of linear, nonlinear, and implicit limit state functions found in the literature.

The transformation methods are interrupted when they encounter a point whose stationarity measure value is sufficiently small. In this way, the conditions are established according to equation 11, using $\epsilon = 10^{-6}$. The method execution is interrupted if these conditions are not met until the code reaches 100 iterations.

In implementing Monte Carlo simulation, MCBF works well for a simulation number up to $9 \times 10^7$ . The MCIS uses a predetermined point as sampling, using the design point obtained through the FORM iHLRF method.

The probability distributions, the moments (mean and standard deviation) of the random variables, and the limit state function $g(x)$ are presented for each problem analyzed

**Problem 1** - Beck [5]: Limit state function $g(x) = x_1 x_2 - x_3$ the variables follow normal probability distribution, with mean values $\mu_{x_1} = 40$, $\mu_{x_2} = 50$ and $\mu_{x_3} = 1000$ and standard deviations $\sigma_{x_1} = 5$, $\sigma_{x_2} = 2.5$ and $\sigma_{x_3} = 200$.

**Problem 2** - Shayanfar [12]: Limit state function $g(x) = x_1^3 + x_1^2 x_2 + x_2^3 - 18$, the variables follow normal probability distribution, with parameters $\mu_{x_1} = 10$, $\mu_{x_2} = 9{,}9$ and $\sigma_{x_1} = \sigma_{x_2} = 5$.

**Problem 3** - Keshtegar e Chakraborty [13]: Limit state function $g(x) = x_1^4 + x_2^2 - 50$, the random variables, $x_1$ has lognormal probability distribution with parameters $\mu_{x_1} = 5$ and $\sigma_{x_1} = 1$, $x_2$ follows Gumbel distribution with parameters $\mu_{x_1} = 10$ and $\sigma_{x_1} = 10$.

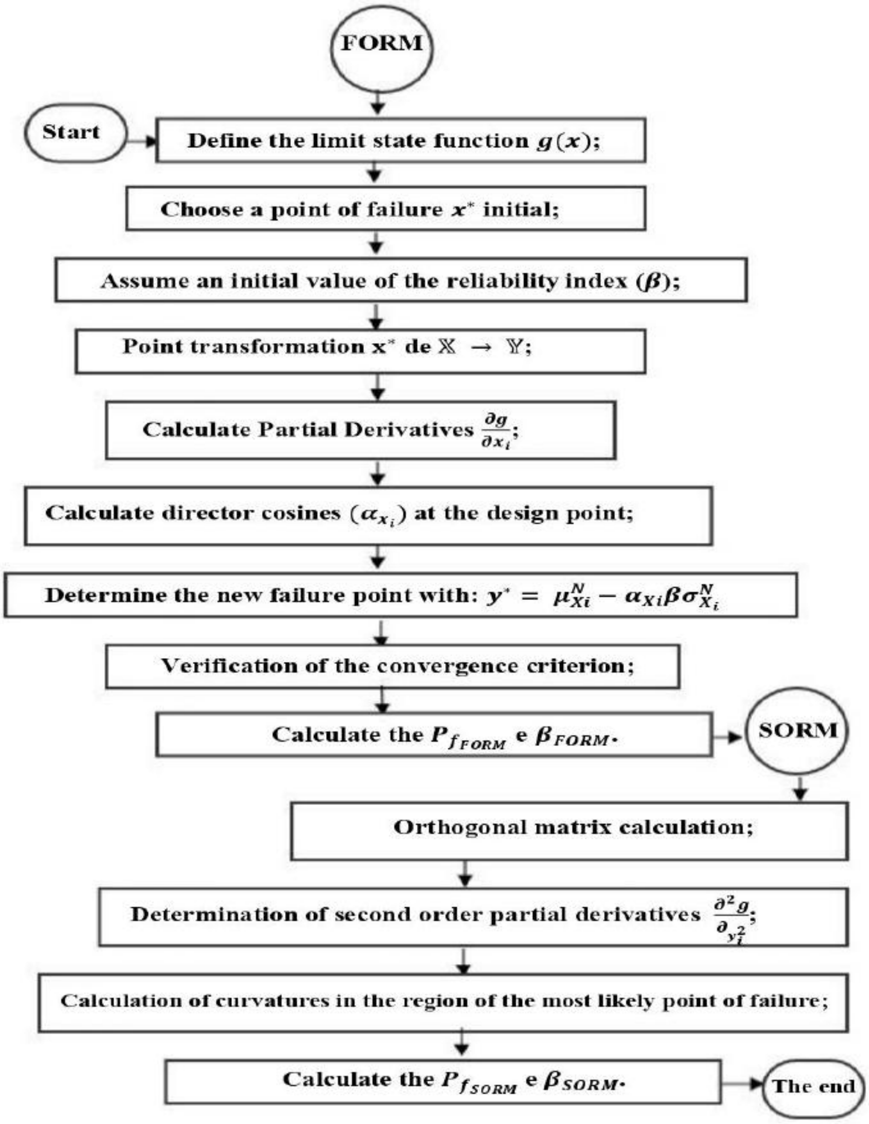


*Fig. 8: Algorithm of the FORM and SORM methods (ALBUQUERQUE [7])*

**Problem 4 -** Li e Zhang [14]: Limit state function $g(A, P) = 3.0 - u_4(A, P)$, the random variables, $x_1$ has lognormal probability distribution with parameters $\mu_{x_1} = 0.0025$ and $\sigma_{x_1} = 0.00025$, $x_2$ follows Gumbel distribution with parameters $\mu_{x_1} = 200$ and $\sigma_{x_1} = 50$. The authors present an analysis of the plane truss with the dimensions and loads shown in Figure 9. The bars' material has a linear elastic behavior, with a modulus of elasticity *E* = 200 GPa. The cross-sections of the truss members have an area of $25\ cm^2$. The structure is subjected to loads *P* = 200 kN, applied vertically at nodes 2 and 4.

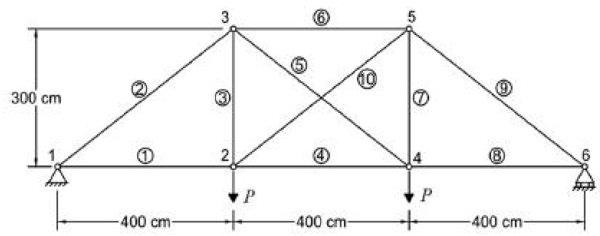


*Fig. 9: Structural scheme of the flat truss (Li e Zhang [14])*

**Problem 5** [Ang e Tang (2007)]: Limit state function $g(R, S) = R - G - Q - W$, the variables have a normal probability distribution, with parameters

$$\mu_R = 975,\ \mu_G = 200,\ \mu_Q = 300,\ \mu_W = 150,\ \sigma_R = 146.25,\ \sigma_G = 14,\ \sigma_Q = 36\ and\ \sigma_W = 30.$$

**Problem 6** - Ang e Tang [15]: In this example, the same data from Problem 5 are used, with changes in the distributions of the variables, *R* and G have lognormal and normal probability distributions, respectively, *Q* and *W* follow the Gumbel probability distribution.

**Problem 7** - GHALEHNOVI *et al*. [16]: Limit state function $g(x) = exp\,(0.2x_1 + 1.4) - x_2$, the variables follow normal probability distribution, with parameters $\mu_{x_1} = \mu_{x_2} = 0$ and $\sigma_{x_1} = \sigma_{x_2} = 1$. This function is weakly nonlinear and its behavior is illustrated in Figure 10.

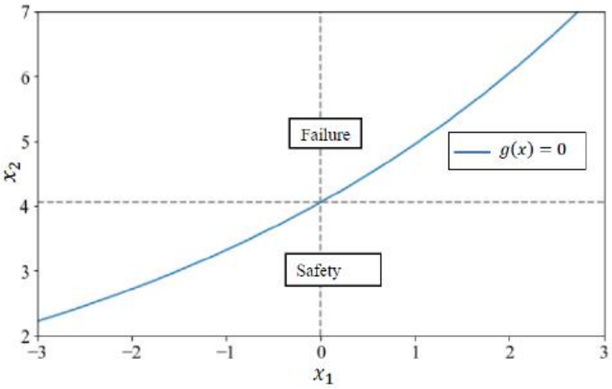


*Fig. 10: Failure threshold and failure and safety domains, for problem 7 (ALBUQUERQUE [7])*

**Problem 8** - SANTOS [6]: Limit state function $g(p,q) = 3 - q + (4p)^4$, the variables follow normal probability distribution, with parameters $\mu_p = \mu_q = 0$ and $\sigma_p = \sigma_q = 1$. This function is strongly nonlinear and its behavior is illustrated in Figure 11.

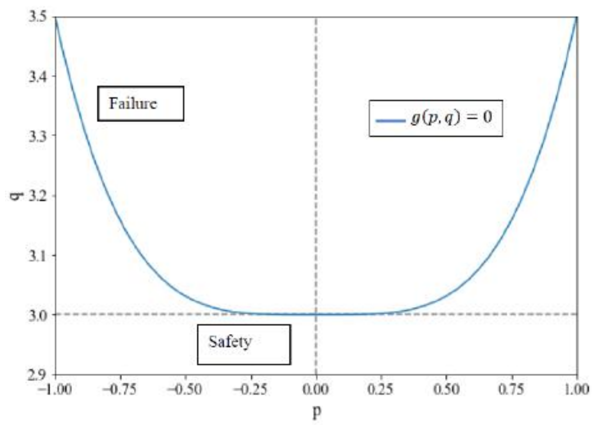


Fig. 11: Failure threshold and the failure and safety domains, for problem 8 (ALBUQUERQUE [7])

**Problem 9** - Beck [5]: Limit state function $g(x) = x_1 + 2x_2 + 3x_3 + x_4 - 5x_5x_6$, the variables follow lognormal probability distribution, with mean values $\mu_{x_1} = \mu_{x_2} = \mu_{x_3} = \mu_{x_4} = 120$, $\mu_{x_5} = 50$ and $\mu_{x_6} = 40$ and standard deviations $\sigma_{x_1} = \sigma_{x_2} = \sigma_{x_3} = \sigma_{x_4} = \sigma_{x_6} = 12$ and $\sigma_{x_5} = 15$.

**Problem 10** – GROOTEMAN [17]: Limit state function $g(x) = 2.5 - 0.2357(x_1 - x_2) + 0.0046(x_1 + x_2 - 20)^4$, the variables have a normal probability distribution, with parameters $\mu_{x_1} = \mu_{x_2} = 10$ and $\sigma_{x_1} = \sigma_{x_2} = 3$.

**Problem 11** - SANTOS *et al.* [18]: Limit state function $g(x) = x_1x_2 - 78{,}12x_3$, the random variables, $x_1$ and $x_2$ have lognormal probability distribution with parameters $\mu_{x_1} = 2\,x\,10^7$, $\mu_{x_2} = 10^{-4}$, $\sigma_{x_1} = 0.5\,x\,10^7$ and $\sigma_{x_2} = 0.2\,x\,10^{-4}$, $x_3$ follows Gumbel distribution with parameters $\mu_{x_3} = 4$ and $\sigma_{x_3} = 1$.

**Problem 12** - SANTOS *et al.* [18]: Limit state function $g(x) = x_1^3 + x_2^3 - 18$, the variables follow normal probability distribution, with parameters $\mu_{x_1} = \mu_{x_2} = 10$ and $\sigma_{x_1} = \sigma_{x_2} = 5$.

**Problem 13** SANTOS *et al.* [18]: Limit state function $g(x) = x_1^3 + x_2^3 - 18$, the variables have a normal probability distribution, with mean values $\mu_{x_1} = 10$ and $\mu_{x_2} = 9{,}9$, and standard deviations $\sigma_{x_1} = \sigma_{x_2} = 5$.

**Problem 14** - Choi *et al.* [19]: Limit state function $g(x) = x_1^4 + 2x_2^4 - 20$, the variables have a normal probability distribution, with parameters $\mu_{x_1} = \mu_{x_2} = 10$ and $\sigma_{x_1} = \sigma_{x_2} = 5$.

**Problem 15** - GHALEHNOVI *et al*. [16]: Limit state function $g(x) = \frac{1}{40}{x_1}^4 + 2x_2^2 + x_3 + 3$, the variables have a normal probability distribution, with parameters $\mu_{x_1} = \mu_{x_2} = \mu_{x_3} = 0$ and $\sigma_{x_1} = \sigma_{x_2} = \sigma_{x_3} = 1$.

**Problem 16 -** [KESHTEGAR e MENG, 2017]: Limit state function

$$g(x) = 0.03 - \left({ql^2}/{2}\right).\left({3.81}/{A_cE_c}\right) + \left({1.13}/{A_sE_s}\right),$$

the random variables, $q, l, A_s$ $and$ $A_c$ have normal probability distribution with parameters $\mu_q = 20000$, $\sigma_q = 1400$, $\mu_l = 12$, $\sigma_l = 0.12$, $\mu_{A_s} = 9.82\,x\,10^{-4}$, $\sigma_{A_s} = 5.9852\,x\,10^{-5}$, $\mu_{A_c} = 0.04$, $\sigma_{A_c} = 0.0048$, $E_s$ follows lognormal distribution with parameters, $\mu_{E_s} = 1.0\,x\,10^{11}$, $\sigma_{E_s} = 6.0\,x\,10^9$ and $E_c$ has Gumbel distribution with parameters, $\mu_{E_c} = 2.0\,x\,10^{10}$ and $\sigma_{E_c} = 1.2\,x\,10^9$. In this example, a roof truss under uniform load is analyzed, shown in Figure 12. The upper chords and compression bars of the truss are made of reinforced concrete, and the bottom chords and tension bars are made of steel.

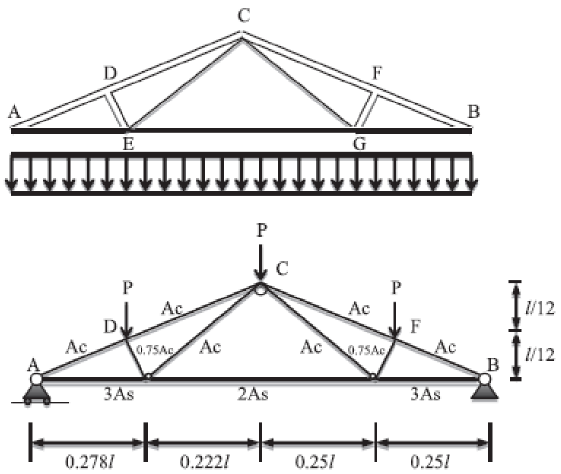


Fig. 12: Vista esquemática da treliça de telhado (KESHTEGAR e MENG [20])

**Problem 17 -** [SHAYANFAR *et al.*, 2017]: Limit state function $g(A_1, A_2, A_3, B, P, E) = 0.1 - B\ .\ u\,(A_1, A_2, A_3, P, E)$, the random variables, $A_1, A_2, A_3\ e\ B$ have normal probability distribution with parameters $\mu_{A_1} = 10^{-2}$, $\sigma_{A_1} = 5.0\ x\ 10^{-4}$, $\mu_{A_2} = 1.5\ x\ 10^{-2}$, $\sigma_{A_2} = 7.5\ x\ 10^{-5}$, $\mu_{A_3} = 6.0\ x\ 10^{-2}$, $\sigma_{A_3} = 3.0\ x\ 10^{-4}$, $\mu_B = 1.0$, $\sigma_B = 0.1$, $P$ follows lognormal distribution with parameters, $\mu_P = 25.0\ x\ 10^5$, $\sigma_P = 2.5\ x\ 10^4$ and $E$ has Gumbel distribution with parameters, $\mu_E = 6.9\ x\ 10^4$ and $\sigma_E = 3.45\ x\ 10^3$. In this problem, a 10-bar truss structure is presented in Figure 13 with three different cross-sectional areas $A_1$ (vertical), $A_2$ (horizontal) e $A_3$ (diagonal). The structure is subjected to two external horizontal loads *P*. The failure of the structure is considered when the horizontal displacement in the upper right corner exceeds the allowable displacement $d_0 = 0.1\ m$. is the modulus of elasticity. The random variable *B* was introduced for the model uncertainties and $L = 9.0$ m is a deterministic variable. it is assumed that $A_1$, $A_2$ , $A_3$, *B*, *P*, *E* are statistically independent random variables.

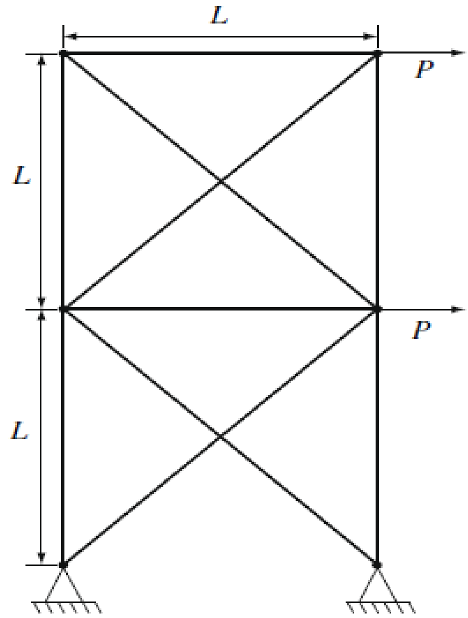


*Fig. 13: 10-bar truss structure (SHAYANFAR et al. [12])*

**Problem 18** - ROUDAK *et al.* [21]: Limit state function $g(x) = \left({}^{7}/_{360}\right) - 360\left({}^{x_1}/_{x_2 x_3{}^4}\right)$, the random variables, $x_1$, $x_2$ e $x_3$ follow normal probability distribution with parameters $\mu_{x_1} = 10.0$, $\sigma_{x_1} = 0.4$, $\mu_{x_2} = 2.0\ x\ 10^7$, $\sigma_{x_2} = 0.5\ x\ 10^7$, $\mu_{x_3} = 0.4$ e $\sigma_{x_3} = 0.01$. In this problem, a continuous beam with three spans is studied, considering the parameters uniform load intensity, modulus of elasticity, and section height. Figure 14 illustrates the analyzed beam.

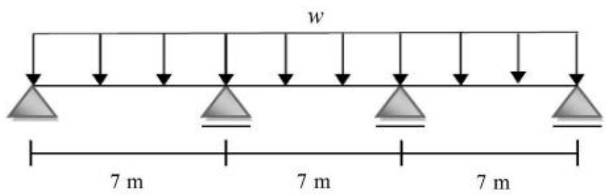


*Fig. 14: Three-span continuous beam (ROUDAK et al. [21])*

Table 1 shows the results of the reliability indices of each method studied for each problem.

*Table 1 - Results of the reliability indices of the analyzed problems*

| Problems | Methods | | | | | |
|---|---|---|---|---|---|---|
| | MCBF | MCIS | FOSM | FORM HLRF | FORM iHLRF | SORM |
| 1 | 3.04 | 3.04 | 3.04 | 3.04 | 3.04 | 3.04 |
| 2 | 2.52 | 2.52 | 1.54 | 1.54 | 2.30 | 2.62 |
| 3 | 3.56 | 3.56 | 1.51 | 1.84 | 3.26 | 3.53 |
| 4 | 3.20 | 3.20 | 4.12 | 3.19 | 3.19 | 3.19 |
| 5 | 2.10 | 2.10 | 2.10 | 2.10 | 2.10 | 2.10 |
| 6 | 2.35 | 2.35 | 2.10 | 2.45 | 2.45 | 2.45 |
| 7 | 3.38 | 3.38 | 3.35 | 3.35 | 3.35 | 3.38 |
| 8 | 3.57 | 3.57 | 3.00 | 3.00 | 3.00 | 3.00 |
| 9 | 2.94 | 2.94 | 3.65 | 3.04 | 3.04 | 3.04 |
| 10 | 2.75 | 2.75 | 2.50 | 2.50 | 2.50 | 2.50 |
| 11 | 4.43 | 4.43 | 3.29 | 4.43 | 4.43 | 4.43 |
| 12 | 2.55 | 2.55 | 2.24 | 2.24 | 2.24 | 2.63 |
| 13 | 2.55 | 2.55 | 1.16 | 1.16 | 2.22 | 2.63 |
| 14 | 2.92 | 2.92 | 0.93 | 0.93 | 2.36 | 2.84 |
| 15 | 3.43 | 3.43 | 2.52 | 2.52 | 3.00 | 3.37 |
| 16 | 2.35 | 2.35 | 0.18 | 0.59 | 2.42 | 2.42 |
| 17 | 4.30 | 4.30 | 5.49 | 3.99 | 4.30 | 4.30 |
| 18 | 2.52 | 2.52 | not converged | not converged | 2.52 | 2.52 |

According to Table 1, problems 1 and 5 tend to converge to the same point, represented by the reliability index. As problems 1 and 5 are characterized by a linear limit state function and normal probability distribution, the application of the methods results in the exact solution of the problems.

From the results obtained in problems 2, 3, 13, 14, and 15, it can be stated that the FOSM and FORM HLRF methods do not converge to stable solutions for highly nonlinear limit state functions, with a significant difference to the reference (MCBF). In these cases, the FORM iHLRF improves the result by decreasing the difference to the reliability index reference but still cannot converge to the results of the MCBF, MCIS, and SORM methods. The high nonlinearity of the functions is the reason for this difference between the methods.

In problem 4, the MCBF method was used with a sample of $10^6$ elements and a computational time of 16 min 35 s and MCIS with $10^4$ simulations with a time of 9 s, and the FOSM, FORM HLRF, FORM iHLRF, and SORM methods with a time smaller than 1 s the reliability indices of Table 1 were determined. The result of the FOSM method is different from the other methods as it does not take into account mixed distributions. The MCIS showed a better performance for this problem, both in computational time and the number of simulations. The Monte Carlo, FORM, and SORM simulation methods obtained results very close to the reference, with a difference of less than 1%. Thus, it can be said that the methods MCBF, MCIS, FORM HLRF, FORM iHLRF, and SORM adequately represented the values found by the authors since the results were very close.

In problems 6 and 11, only the FOSM method did not converge to the result of the other methods because the distributions are mixed. Given the weak nonlinearity of the limit state function, the FORM HLRF and FORM iHLRF methods present identical behavior to the MCBF, MCIS, and SORM methods.

Regarding the results of problem 7, there is a difference in the values of the reliability index of the

FOSM, FORM HLRF, and FORM iHLRF methods; however, the differences found are approximately 1% compared to the other methods. This difference presents a relatively small divergence. Thus, it can be affirmed that this function is weakly nonlinear and that the analyzed methods effectively determine the reliability index of this function.

In problems 8 and 10, the MCBF was used with a sample with $10^6$ elements and MCIS with $10^4$ elements. Despite presenting the same results, the transformation methods converge to a result far from the reference (MCBF). In these problems, the second-order curvatures are zero. A surface of zero curvature is a plane, so FORM and SORM got the same results.

It can be noted that the results obtained in problem 9 produced very similar results, except for the FOSM method. Only the FOSM method did not converge to the result of the other transformation methods because it does not converge correctly for non-normal distributions and still obtained a β greater than the other methods. This lack of convergence is because this function is weakly nonlinear, and if the other methods had a normal probability distribution, they would converge to the same result.

In problem 12, the difference between the reliability indices of the Monte Carlo methods compared to the FOSM, FORM HLRF, and FORM HLRF is 12%. The MCIS once again proved to be more efficient in the number of simulations ($10^4$ elements) than the MCBF ($10^6$ elements).

The MCBF method was used with a sample with $10^7$ elements, MCIS with $10^4$ elements. The analysis of problems 16 and 18 was performed considering the MCBF method as a reference. SORM and FORM iHLRF approximate MCBF and MCIS. There are very marked differences between the results of the FOSM and FORM HLRF methods with 92% and 78%, respectively, in problem 16 to the MCBF reliability index. In problem 18, the FOSM and FORM HLRF methods failed to converge to the result. The high nonlinearity is the reason for this difference.

The truss presented in problem 17 is a well-known benchmark example studied by some researchers. And using the MCBF method with a sample with elements, it obtained a computational time of 11h 45min 22 s, MCIS with $9x10^7$ simulations with a time of 15 s, and the FOSM, FORM HLRF, FORM iHLRF, and SORM methods with a time of less than 5 s. The analysis of the results was performed considering the MCBF method as a reference. The MCIS method showed once again that it improves the performance and efficiency of the MCBF regarding the number of simulations and computational time. As expected, FOSM did not converge to the benchmark result. And the FORM HLRF walked 7% away from the target reliability index.

Figure 15 compares the reliability index of the 18 problems analyzed.

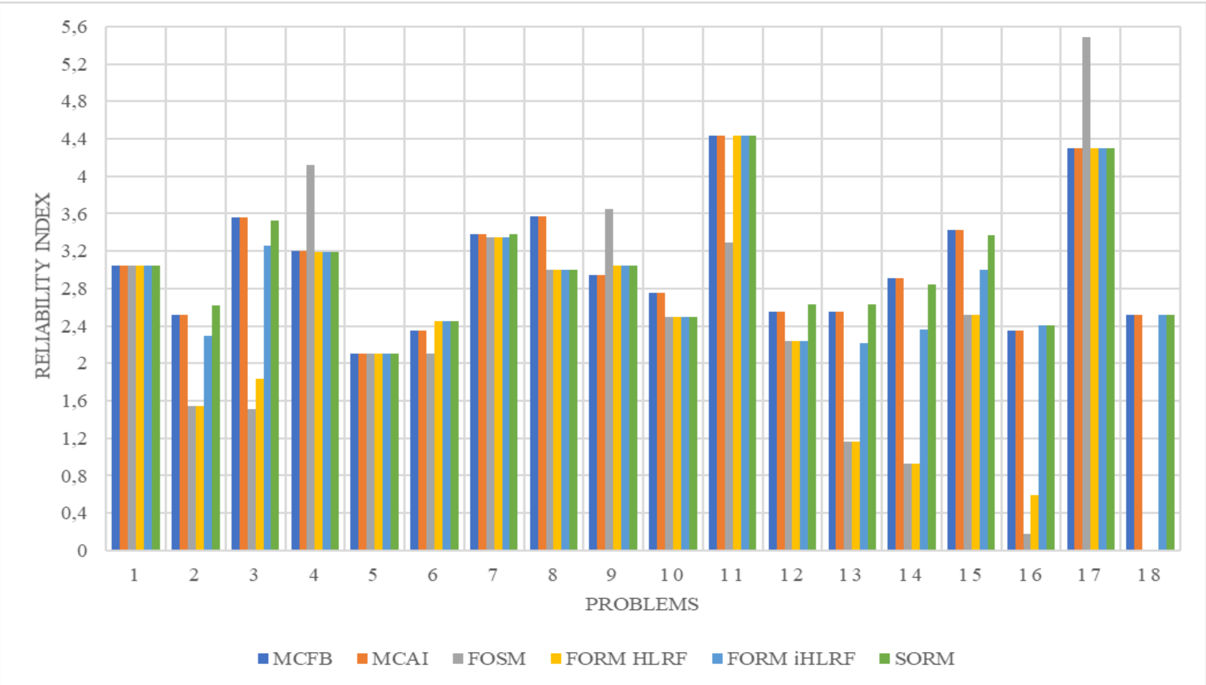


*Fig. 15: Comparison of the reliability index of the analyzed examples.*

It is observed in Figure 15 that the SORM method proved to be adequate for the probabilistic analysis and returned values equal to or very close to the MCBF and MCIS methods in the analyzed problems. SORM is calculated by obtaining the reliability index of the FORM method. In this way, SORM becomes more advantageous because it improves the FORM result by including additional information about the curvature of the limit state. In the case of problems 8 and 10, the curvatures in the failure region are null despite the function being nonlinear.

Also, according to Figure 15, it is observed that the MCBF and MCIS methods successfully solved 100% of the examples, SORM solved 89%, FOSM, FORM HLRF, and FORM iHLRF solved 11%, 44%, and 56%, respectively. So, of the transformation methods, SORM successfully solved more examples than the FOSM and FORM methods.

Regarding the performance of the FOSM and FORM methods, the number of iterations is considered a criterion. FOSM and FORM HLRF reached the maximum number of iterations ($k$ = 100) in solving problems 11, 13, 14, 15, and 16, and FORM iHLRF reached 56 as the maximum number of iterations in problem 14. Therefore, there is a significant difference based on this performance criterion of the FORM iHLRF against the FORM HLRF.

## IV. CONCLUSION

This work presented a comparative study of structural reliability methods (MCBF, MCIS, FOSM, FORM HLRF, FORM iHLRF, and SORM), taking into account the results of the reliability index. The performance of the methods implemented in the Python language was verified through eighteen linear, nonlinear and implicit limit state function problems.

The MCBF and MCIS simulation methods worked well for all analyzed examples. The MCIS method presented the same precision as the MCBF but with greater efficiency, converging faster by using a smaller number of simulations and with a much lower computational cost in analyzing the implicit limit state functions.

It was observed that the use of the FOSM method is quite limited, as it presents results far from the references. Thus, it can be said that the technique is not suitable for most of the analyzed examples, as it only uses moments of first and second order, not taking into account the form of the probability distribution. If the basic random variables are not normal and the limit state function is linear, the imprecision level grows.

The FORM HLRF and FORM iHLRF methods did not obtain good results for the highly nonlinear functions analyzed. Although these methods transform distributions that are not normally distributed into equivalent normals, they work with a linearization of the failure surface. If the function is highly nonlinear around the point of failure, the results of this method are not as good as the MCBF and MCIS methods. The FORM iHLRF is closer to the reference values for the analyzed functions. Therefore, the iHLRF algorithm improves the HLRF through a linear search for the design point.

The SORM presented the same or very close results concerning the reference (MCBF). The application of this method proved to be efficient in sixteen of the eighteen problems analyzed and carried out the analysis of second-order structural reliability for the functions effectively, optimizing in a relevant way the results obtained by the FORM iHLRF method. SORM is more beneficial because it improves the FORM result by including additional information about the limit state curvature at the design point. In problems 8 and 10, the second-order curvatures are zero. A surface of zero curvature is a plane; FORM and SORM gave the same results.


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## REFERENCES


[1] HALDAR, A.; MAHADEVAN, S. 2000. Reliability assessment using stochastic finite element analysis. John Wiley & Sons.

[2] REAL, M. V.; 2000. CAMPOS FILHO, A.; MAESTRINI, S. R. 2003. Response variability in reinforced concrete structures with uncertain geometrical and material properties. Nuclear Engineering and Design, v. 206, p. 205-220. https://doi.org/10.1016/S0029-5493(03)00110-9

[3] ALBUQUERQUE, C. S.; REAL, M. V. (2020). Estudo comparativo de métodos de confiabilidade estrutural aplicados a estruturas de concreto, 62º Congresso Brasileiro do concreto. https://doi.org/10.4322/2175-8182.62CBC005

[4] VERMA, A. K.; AJIT, S.; DURGA, R. K. (2016). Reliability and Safety Engineering. – 2. ed. – Springer-Verlag London Ltd.

[5] BECK, A. T. 2019. Confiabilidade e Segurança das Estruturas – 1. ed. – Rio de Janeiro: Elsevier.

[6] SANTOS, K. R. M. dos. 2014. Técnicas de amostragem inteligente em simulação de Monte Carlo. Dissertação de

Mestrado - Universidade de São Paulo - Escola de Engenharia São Carlos, São Paulo.

[7] ALBUQUERQUE, C. S. 2021. Estudo comparativo de métodos computacionais de avaliação de confiabilidade estrutural. Dissertação de Mestrado - Universidade Federal do Rio Grande, Rio Grande -RS.

[8] MELCHERS, R. E; BECK, A. T. 2018. Structural Reliability Analysis and Prediction. John Wiley & Sons.

[9] JOINT COMMITTEE ON STRUCTURAL SAFETY, JCSS: Probabilistic Model Code, 2001.

[10] SUDRET, B.; DER KIUREGHIAN A. 2000. Stochastic Finite Element Methods and Reliability: A state-of-the-art Report. Research report No. UCB/SEMM-2000/08, Dep. of Civil & Env. Eng., University of California, Berkeley.

[11] CORDEIRO, A. L. R. 2009. Aplicação dos métodos FORM e SORM para avaliação de dutos corroídos. Dissertaçãode Mestrado, Pontifícia Universidade Católica do Paraná, Curitiba.

[12] SHAYANFAR, M. A.; BARKHORDARI, M. A.; ROUDAK, M. A. 2017. An efficient reliability algorithm for locating design point using the combination of importance sampling concepts and response surface, Communications in Nonlinear Science and Numerical Simulation. (47) 223–237. https://doi.org/10.1016/j.cnsns.2016.11.021

[13] KESHTEGAR, B.; CHAKRABORTY, S. 2018. A hybrid self-adaptive conjugate Þrst order reliability method for robust structural reliability analysis. Applied Mathematical Modelling, 53: 319-333. https://doi.org/10.1016/j.apm.2017.09.017

[14] LI, Y.; D. Zhang, D. 2013. Plane Truss Reliability Numerical Simulation Based on MATLAB. Applied Mechanics and Materials, Vols. 256-259, p. 1091-1096. https://doi.org/10.4028/www.scientific.net/AMM.256-259.1091

[15] ANG, A. H-S.; TANG, W. H. 2007. Probability Concepts in Engineering: Emphasis on Applications to Civil and Environmental Engineering. – 2. ed. – John Wiley & Sons.

[16] GHALEHNOVI, M.; RASHKI, M.; AMERYAN, A. 2020. First order control variates algorithm for reliability analysis of engineering structures. Applied Mathematical Modelling, v. 77, p. 829–847. https://doi.org/10.1016/j.apm.2019.07.049

[17] GROOTEMAN, F. 2008. Adaptive radial-based importance sampling method for structural reliability. Structural Safety, 30, 533-542. https://doi.org/10.1016/j.strusafe.2007.10.002

[18] SANTOS, S. R.; MATIOLI, L. C.; BECK, A. T. 2012. New optimization algorithms for structural reliability analysis. Computer Modeling in Engineering and Science, v. 83, p. 23–56. https://doi:10.3970/cmes.2012.083.023

[19] CHOI, S. K.; GRANDHI, R. V.; CANFIELD, R. A. 2007. Reliability-based Structural Design**.** Springer-Verlag London Limited.

[20] KESHTEGAR, B.; MENG, Z. 2017. A hybrid relaxed first-order reliability method for efficient structural reliability analysis, Structural Safety. 66, 84–93. https://doi.org/10.1016/j.strusafe.2017.02.005

[21] ROUDAK, M.; SHAYANFAR, M.; BARKHORDARI, M.; KARAMLOO, M. 2017. A robust approximation method for nonlinear cases of structural reliability analysis, International Journal of Mechanical Sciences. 133, 11-20. https://doi.org/10.1016/j.ijmecsci.2017.08.038