New bar finite element for modeling massive columns with linearly variable rectangular section

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Abstract**—***In this article, the stiffness matrix of a column with linearly variable rectangular cross section along the longitudinal axis is obtained. As it is well known in structural analysis, the stiffness matrix can be obtained by inversion of the flexibility matrix. So, the terms of this matrix for variable section beam are obtained via the energetic method of the principle of virtual works. Parameters called $\alpha_i, \alpha_f, \beta_f$ and ε are obtained and they are valid only for sections with variable dimensions. The stiffness matrix is then obtained in function of these parameters, from de inversion of the flexibility matrix.Finally, the modal analysis of the abutment is carried out for the case of the support of a bridge 100 meters high, whose numerical validation of the new bar finite element is performed by means of an exact solution by the Continuous Medium Technique (CMT) and by modeled finite elements in Ansys academic version.*

Keywords**—*****New Bar Finite Element, Massive Columns, Stiffness Matrix, Vibration modes.***

I. INTRODUCTION

In the structural analysis, the Method of Displacements is relevant, which consists of resolving the structure by obtaining the initial deformations and, following these deformations, it is necessary to draw the diagrams of the internal forces of the beam that make up the structure (SUSSEKIND, 1978) [1]. The incognitos of the problem are: the angle of rotation and the linear displacements at each node to be properly constrained from the displacements. This should be considered for the various columns.

As illustrated in Kassimali [2], the advantage of using the Method of Displacements over the Method of Forces is based on the fact that there is no difficulty in choosing the incognitos, since the fundamental problem is only one per structure. Thus, the definition of hypergeometric grades emerges, which is the number of displacements (linear and angular) that can occur at the nodes of a given structure. For a high-length Bridge Pillar, it is economically relevant to adopt a variable cross section, with a more robust base than the top.

Thus, by adding such a taper in the column shaft it is constructively more feasible that such a variation is linear, which motivates this article with exact attainment of the Stiffness Matrix of a beam element with rectangular linearly variable cross section along the axial axis *x*. In the field of approximate solutions in such determination, Luo et al. [3] and Brown [4] are quoted.Thus formulating a New Bar Finite Element.

II. NEW BAR FINITE ELEMENT FOR LINEARLY VARIABLE RECTANGULAR SECTION

a) Method of Forces

Also called Flexibility Method and developed by James Clerk Maxwell in 1864, in order to determine deflections in trusses with beam links by labeled joints. To this end, the internal energy activated only by the normal effort was used [5]. Still, based on Timoshenko [5], when promoting the temporal delimitation of contributions in structural analysis in 1874, Otto Mohr's technique arises in response to the problem of nodes (beam joining) of trusses formed by two or more rivets,making the connections no longer labeled and, therefore, the need to compute the deformation internal force energy with all the internal forces. In equation 1 (a), the calculation of deflections is verified by the Maxwell formulation for labeled trusses. On the other hand, in equation 1 (b), the same occurs, however, for connections with two or more fixations per node, according to Mohr's formulation.

$$\delta_{i0} = \int_{\Omega} \frac{(N_i.N_0)}{E.A}\, dx \qquad (1\,a)$$

$$\delta_{i0} = \int_{\Omega} \frac{(N_i.N_0)}{E.A}\, dx + \int_{\Omega} \frac{(M_i.M_0)}{E.I}\, dx + \int_{\Omega} k_c.\frac{(V_i.V_0)}{E.A}\, dx \qquad (1\,b)$$

with: δ_{i0} = Nodal deflexion to the i node; N_i, M_i, V_i = Axial load, Flexion moment, Bending moment and Shear force for the real load of the structure; and N_0, M_0, V_0 = Axial load, Bending moment and Shear force for the virtual load via P.V.W. acting on Node *i*.

Yet, in Charlton [6], Maxwell´s publication entitled: “On reciprocal figures, frames and diagrams of forces” it is evident. In the work of the same author, page 83, Mohr´s publication entitled: " Beitrag Zur Theorie Du Bogenfachwerk träger " can be seen. In 1886, Heinrich Müller - Breslau postulated the systematization of the Method of Forces defined earlier by Maxwell and Mohr (KINNEY, 1957) [7]. The publication of such systematization is: "Die neue methoden der baukonstruktionen". The fundamental basis of the Method of Forces is the compatibilization of the angular and linear displacements of the extracted connections in order to establish extra equations and make the system of equations of the problem linearly independent and, therefore, make it solvable.

The compatibilization of the displacements due to the connections extracted via definition of said method will be adopted from the positive convention for linear displacements in the positive direction of the *x* and *y* axes and rotation with vector notation in the same direction as the positive *z* axis. Thus, using a formulation present in Kiseliov [8], the system of displacement compatibility equations is written in matrix notation as:

$$\{d\} = [F].\{X\} + \{\delta\} + \{\delta^T\} \qquad (2)$$

with: $\{d\}$ = Vector of displacements in the initial hyperstatic structure; $\{\delta\}$ = Vector of displacements in the fundamental problem; $\{\delta^T\}$ = Vector of displacements in the thermal problem; $\{X\}$ = Vector of incognitos for the Method of Forces; and $[F]$ = Flexibility Matrix.

In order to assemble the Flexibility Matrix $[F]$. just consider the derived systems to become, in the background, both the fundamental problem responsible for the analysis of the actual and active load in the initial structure and the thermal problem. In addition, the vector {δ} is neglected in this analysis, since it is only desired to obtain the Flexibility Matrix. Such Systems derived from a fixed-fixed beam are shown in Figure 1 as much as the Main System.

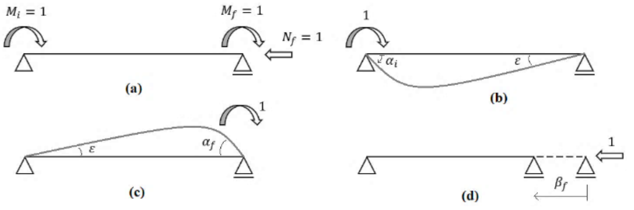

Fig. 1: Method of Forces: (a) Main System, (b) 1st. Derived System, (c) 2nd. Derived System, (d) 3rd. Derived System

At the initial node there is the Degree of Freedom θ_i. The final node is characterized by the degree of freedom also in rotation θ_f and, finally, δ_f in linear displacement for the final node. The system of equations is expressed by:

$$\begin{Bmatrix} \theta_i \\ \theta_f \\ \delta_f \end{Bmatrix} = \begin{bmatrix} \alpha_i & \varepsilon & 0 \\ \varepsilon & \alpha_f & 0 \\ 0 & 0 & \beta_f \end{bmatrix} . \begin{Bmatrix} M_i \\ M_f \\ N_f \end{Bmatrix} \qquad (3)$$

with: α_i, α_f = Angular displacement for the initial and final node of the beam, respectively; ε = Angular displacement in a node contrary to where α_i and α_f occur; M_i, M_f = Flexion moment along the beam with unitary moment imposition; V_i, V_f = Shear force along the beam with unitary and vertical load imposition; and N_f = Axial load of the beam with unitary and horizontal load imposition.

b) Obtaining the displacements α, β and ε via PVW

The Principle of Virtual Works (PVW) was postulated by John Bernoulli in 1717 and is based on the Principle of Energy Conservation. This principle was also linked to the concept of virtual displacement. In virtual displacement, when the material point is in equilibrium, real displacement cannot occur. And for a particle to be in equilibrium, the condition of nullity to the work of all external forces must be satisfied (STAMATO, 1983) [9]. Figure 2 shows the linear and angular displacements β_f, andα_i, α_f and ε, respectively, as:

$$\alpha_i = \int_0^L \frac{(M_i.\bar{M}_i)}{E.I_z(x)}\, dx + \int_0^L k_c.\frac{(V_i.\bar{V}_i)}{G.A(x)}\, dx \qquad (4\,a)$$

$$\alpha_f = \int_0^L \frac{(M_f.\bar{M}_f)}{E.I_z(x)}\, dx + \int_0^L k_c.\frac{(V_f.\bar{V}_f)}{G.A(x)}\, dx \qquad (4\,b)$$

$$\varepsilon = \int_0^L \frac{(M_i.M_f)}{E.I_z(x)}\, dx + \int_0^L k_c.\frac{(V_i.V_f)}{G.A(x)}\, dx \qquad (4\,c)$$

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$$\beta_f = \int_0^L \frac{(N_f.\bar{N}_f)}{E.A(x)}\,dx \tag{4 d}$$

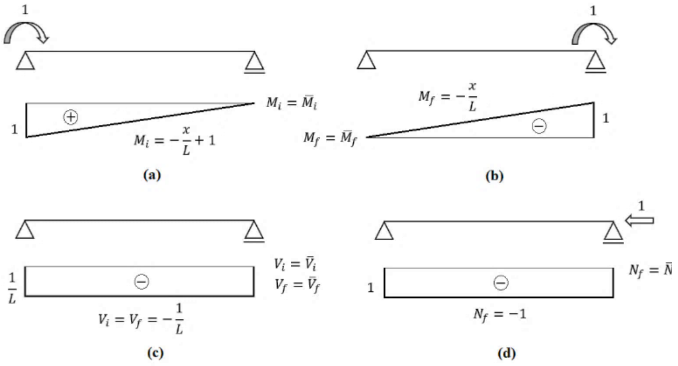
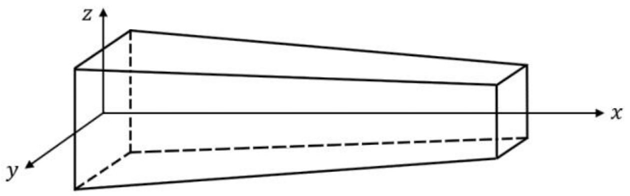


Fig. 2: Internal Force Diagrams: (a)$M_i \equiv \bar{M}_i$, (b) $M_f \equiv \bar{M}_f$, (c) $V \equiv \bar{V}$, (d) $N_f \equiv \bar{N}_f$

For the beam element of length L and dimensions in the cross section $H_y(x)$ and $H_z(x)$, as shown in Figure 3, the cross section area $A\ (x)$ and the Moment of Inertia $I_z(x)$ around the z axis, is written as:



$b_z = H_z(x=0)$ $\quad$ $h_z = H_z(x=L)$

$b_y = H_y(x=0)$ $\quad$ $h_y = H_y(x=L)$

Fig. 3: Beam Element with dimension in the linearly variable cross section along the axial axis x

$$I_z(x) = k_1.x^4 + k_2.x^3 + k_3.x^2 + k_4.x + k_5 \tag{5 a}$$

$$A(x) = k_6.x^2 + k_7.x + k_8 \tag{5 b}$$

where: $k_1 = A.C^3$; $k_2 = C^2.(3.A.D + B.C)$; $k_6 = A.C$; $k_3 = 3.C.D.(A.D + B.C)$; $k_5 = B.D^3$; $k_8 = B.D$; $k_4 = D^2.(A.D + 3.B.C.D)$; $k_7 = A.D + B.C$;

$$B = b_z; D = b_y;\ A = \frac{h_z - b_z}{L};\ \ C = \frac{h_y - b_y}{L}.$$

with: b_z, h_z –Cross section dimensions, parallel to axis z, in the initial and final section, respectively; b_y, h_y –Cross section dimensions, parallel to axis y, in the initial and final section, respectively; E –Longitudinal Elasticity Module; G – Cross Elasticity Module; $I_z(x)$ – Variation along the axial axis x of the inertia moment surrounding axis z; $A(x)$ – Variation along axial axis x of the cross section area; k_c – Shape factor and L – Beam length.

Analysing the Area Static Moment $Q(x)$ for a rectangular section with dimensions $H_y(x)$ and $H_z(x)$, characterized in Figure 4, results in:

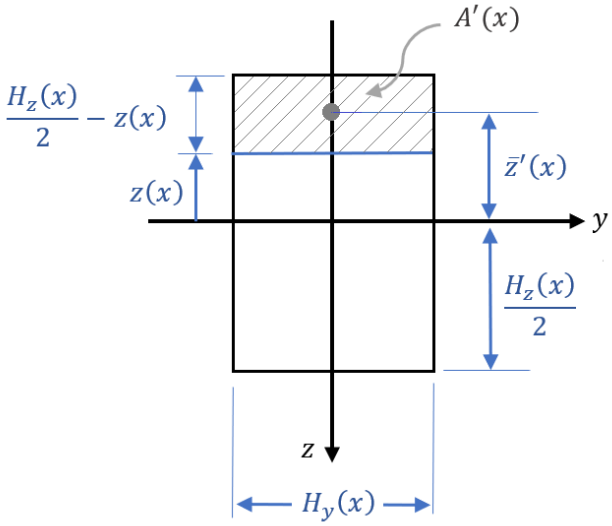


Fig. 4: AreaStaticMoment

$$A'(x) = \left[\frac{H_z(x)}{2} - z(x)\right].H_y(x) \tag{6 a}$$

$$I_z(x) = \frac{H_z(x).H_y^3(x)}{12} \tag{6 b}$$

$$\bar{z}'(x) = \frac{A'(x)}{2.H_y(x)} + z(x) \tag{6 c}$$

$$Q(x) = A'(x).\bar{z}'(x) = \frac{H_y(x)}{2}.\left[\frac{H_z^2(x)}{4} - z^2(x)\right] \tag{6 d}$$

Proceeding the calculation of the shape factor k_c, after transforming the integration in area A into along the length, there is:

$$k_c = \frac{A(x)}{I_z^2(x)}.\int_A \frac{Q^2(x)}{H_y^2(x)}\,dA$$

$$= \frac{A(x)}{I_z^2(x)}.\int_{-\frac{H_z(x)}{2}}^{\frac{H_z(x)}{2}} \left[\frac{Q^2(x)}{H_y^2(x)}.H_y(x)\right] dz \tag{7}$$

When applying equations 6 (a - d) in equation (7) and carrying out the integration in z and consequent simplifications, we conclude that the shape factor $k_c$remains unchanged along axis x with the following value:

$$k_c = \frac{5}{6} \tag{8}$$

Finally, when applying the equations (5), 6 (a - d) and (8) in equations 4 (a - d), the flexibility coefficients α_i, α_f, ε and $\beta_f$are attained, expressed by:

- ***if $A.D \neq B.C$:***

$$\alpha_i = \frac{12}{E.\eta_3}.\left[\eta_1 - \eta_2.\ln\left(\frac{B+A.L}{A}\right) + \eta_2.\ln\left(\frac{B}{A}\right) + \eta_2.\ln\left(\frac{D+C.L}{C}\right) - \eta_2.\ln\left(\frac{D}{C}\right)\right] + \frac{k_c}{G.\eta_4}.\left[\ln\left(\frac{B+A.L}{D+C.L}\right) - \ln\left(\frac{B}{D}\right)\right] \quad (9\,a)$$

$$\alpha_f = \frac{12}{E.\eta_7}.\left[\eta_5 + \eta_6.\ln\left(\frac{B+A.L}{A}\right) - \eta_6.\ln\left(\frac{B}{A}\right) - \eta_6.\ln\left(\frac{D+C.L}{C}\right) + \eta_6.\ln\left(\frac{D}{C}\right)\right] + \frac{k_c}{G.\eta_4}.\left[\ln\left(\frac{B+A.L}{D+C.L}\right) - \ln\left(\frac{B}{D}\right)\right] \quad (9\,b)$$

$$\varepsilon = \frac{12}{E.\eta_{10}}.\left[\eta_8 + \eta_9.\ln\left(\frac{B+A.L}{A}\right) - \eta_9.\ln\left(\frac{B}{A}\right) - \eta_9.\ln\left(\frac{D+C.L}{C}\right) + \eta_9.\ln\left(\frac{D}{C}\right)\right] + \frac{k_c}{G.\eta_4}.\left[\ln\left(\frac{B+A.L}{D+C.L}\right) - \ln\left(\frac{B}{D}\right)\right] \quad (9\,c)$$

$$\beta_f = \frac{1}{E}.\left[\frac{\ln\left(\frac{B+A.L}{D+C.L}\right) - \ln\left(\frac{B}{D}\right)}{(A.D - B.C)}\right] \quad (9\,d)$$

- ***if A.D = B.C:***

$$\alpha_i = \alpha_f = \frac{4.L}{E.B.D^3} + \frac{k_c}{G.B.D.L} \quad (9\,e)$$

$$\varepsilon = \frac{-2.L}{E.B.D^3} + \frac{k_c}{G.B.D.L} \quad (9\,f)$$

$$\beta_f = \frac{L}{E.B.D} \quad (9\,g)$$

where A, B, C and D are parameters linked to geometrical dimensions of the cross section. And yet, list yourself: $\eta_1 = L.(A.D + B.C).(2.B.D + 3.A.D.L - B.C.L)$;

$\eta_3 = -2.D^2.L^2.(A.D + B.C)^3$; $\eta_4 = L^2.(A.D + B.C)$;

$2.\eta_8 = -L.(A.D - B.C).(2.B.D + A.D.L + B.C.L)$;

$\eta_5 = -L.(A.D - B.C).(2.B.D - A.D.L + 3.B.C.L)$;

$\eta_2 = 2.D^2.(A.L + B)^2$; $\eta_6 = 2.B^2.(C.L + D)^2$;

$\eta_7 = 2.L^2.(C.L + D)^2.(A.D - B.C)^3$;

$\eta_{10} = D.L^2.(C.L + D).(A.D + B.C)^3$;

$\eta_9 = B.D.(A.L + B).(C.L + D)$.

c) Method of Displacements

Such method consists of resolving the structure by initially obtaining the deformations by means of the internal forces on the beams that make up the structure, while in the Method of Forces, extra bonds are extracted to structural staticity and stability. In the Method of Displacements, the locking of the bound nodes of the structure is promoted in order to attain the connection of the beams by means of embedments. In matrix notation, the system of equations of equilibria of the unbalancing forces by nodes is expressed as follows:

$$\{M\} = [K].\{D\} + \{\gamma\} + \{\gamma^T\} \quad (10)$$

where: $\{M\}$ = Vector of unbalancing forces; $[K]$ = Stiffness Matrix; $\{D\}$ = Vector of incognitos in the Method of Displacements; $\{\gamma\}$ = Vetor of unbalancing forces in the fundamental problem fundamental; $\{\gamma^T\}$ = Vector of unbalancing forces in the thermal problem; and k_i, k_f, a, r_f = Stiffness coefficients.

Using the definitions in Kassimali[2] and Kiseliov [8], the Stiffness Matrix $[K]$ will be the inverse of the Flexibility Matrix $[F]$. The terms of stiffness are concluded, such as:

$$[K] = [F]^{-1} = \begin{bmatrix} \alpha_i & \varepsilon & 0 \\ \varepsilon & \alpha_f & 0 \\ 0 & 0 & \beta_f \end{bmatrix}^{-1} = \begin{bmatrix} k_i & a & 0 \\ a & k_f & 0 \\ 0 & 0 & r_f \end{bmatrix} \quad (11\,a)$$

$$k_i = \frac{\alpha_f}{\alpha_i.\alpha_f - \varepsilon^2} \quad (11\,b)$$

$$k_f = \frac{\alpha_i}{\alpha_i.\alpha_f - \varepsilon^2} \quad (11\,c)$$

$$a = \frac{-\varepsilon}{\alpha_i.\alpha_f - \varepsilon^2} \quad (11\,d)$$

$$r_f = \frac{1}{\beta_f} \quad (11\,e)$$

The present formulation was proposed by George Alfred Maney in 1915, and was also denominated Rotation-Arrow Method. See [10]; [11] and [12].

III. MATRIX CONDENSATION

Based on Paz [13] and the application in the calculation of natural frequencies in Alves Filho (p. 200) [14], matrix condensation consists of rewriting the system of equations in terms of some of its variables. In Figure 5 shows a pillar with n subdivisions and (n + 1) nodes, as well as the resulting degrees of freedom (δ and θ) and nodal forces, in order to exemplify such a condensation procedure.

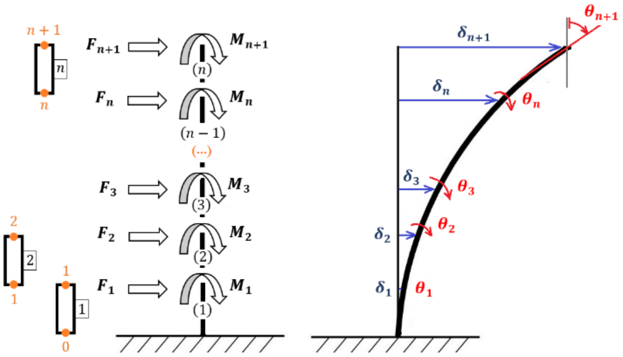


Fig. 5: column subdivision by finite elements of bar

The ODE's system remaining, for the non-damped vibration, expressed by:

$$\begin{bmatrix}[M_{\theta\theta}] & [M_{\theta\delta}] \\ [M_{\delta\theta}] & [M_{\delta\delta}]\end{bmatrix}.\begin{Bmatrix}\{\ddot{\theta}\} \\ \{\ddot{\delta}\}\end{Bmatrix} + \begin{bmatrix}[K_{\theta\theta}] & [K_{\theta\delta}] \\ [K_{\delta\theta}] & [K_{\delta\delta}]\end{bmatrix}.\begin{Bmatrix}\{\theta\} \\ \{\delta\}\end{Bmatrix} = \begin{Bmatrix}\{M\} \\ \{F\}\end{Bmatrix} \qquad (12)$$

with: $\{\theta\}^T = \{\theta_1 \quad \theta_2 \quad \theta_3 \quad \dots \quad \theta_{n-1} \quad \theta_n\}$;

$\{\delta\}^T = \{\delta_1 \quad \delta_2 \quad \delta_3 \quad \dots \quad \delta_{n-1} \quad \delta_n\}$;

$\{M\}^T = \{M_1 \quad M_2 \quad M_3 \quad \dots \quad M_{n-1} \quad M_n\}$;

$\{F\}^T = \{F_1 \quad F_2 \quad F_3 \quad \dots \quad F_{n-1} \quad F_n\}$.

The submatrix$[M_{\theta\theta}]$ presents the terms of rotational masses, with little representativeness (in magnitude) in relation to translational masses (included in the submatrix $[M_{\delta\delta}]$). From this statement, sub-matrices$[M_{\theta\delta}]$ and $[M_{\delta\theta}]$ are also disregarded, and equation is rewritten (12) as:

$$\begin{bmatrix}[0] & [0] \\ [0] & [M_{\delta\delta}]\end{bmatrix}.\begin{Bmatrix}\{\ddot{\theta}\} \\ \{\ddot{\delta}\}\end{Bmatrix} + \begin{bmatrix}[K_{\theta\theta}] & [K_{\theta\delta}] \\ [K_{\delta\theta}] & [K_{\delta\delta}]\end{bmatrix}.\begin{Bmatrix}\{\theta\} \\ \{\delta\}\end{Bmatrix} = \begin{Bmatrix}\{M\} \\ \{F\}\end{Bmatrix} \qquad (13)$$

Expressing equation (13) in the form of a matrix equation system, we have:

$$[K_{\theta\theta}].\{\theta\} + [K_{\theta\delta}].\{\delta\} = \{M\} \qquad (14\ a)$$

$$[M_{\delta\delta}].\{\ddot{\delta}\} + [K_{\delta\theta}].\{\theta\} + [K_{\delta\delta}].\{\delta\} = \{F\} \qquad (14\ b)$$

In order to express the displacements { δ }, the rotation vector { θ } is isolated in equation (14 a) and applied to eq. (14 b), concluding:

$$[M_{\delta\delta}].\{\ddot{\delta}\} + [K_{\delta\theta}].([K_{\theta\theta}]^{-1}.\{M\} - [K_{\theta\theta}]^{-1}.[K_{\theta\delta}].\{\delta\}) + [K_{\delta\delta}].\{\delta\} = \{F\} \qquad (15)$$

rearranging, there is:

$$[M_{\delta\delta}].\{\ddot{\delta}\} + [K^*].\{\delta\} = \{F^*\} \qquad (15\ a)$$

with: $[K^*] = [K_{\delta\delta}] - ([K_{\delta\theta}].[K_{\theta\theta}]^{-1}.[K_{\theta\delta}])$;

$\{F^*\} = \{F\} - ([K_{\delta\theta}].[K_{\theta\theta}]^{-1}.\{M\})$.

and: $[K^*]$ the condensed stiffness matrix. and $\{F^*\}$ the condensed vector of transverse forces.

Condensed ODE for proportionally dampened vibration($[C^*] = \alpha_m.[M_{\delta\delta}] + \alpha_k.[K^*]$)is expressed by:

$$[M_{\delta\delta}].\{\ddot{\delta}\} + [C^*].\{\dot{\delta}\} + [K^*].\{\delta\} = \{F^*\} \qquad (16)$$

IV. VERIFICATION OF (FEM) MODELING VIA (CMT)

Through the dynamic analysis of the pillar by the Continuous Medium Technique (CMT) processed in Melo [15], the differential equation of the problem is written, as:

$$-[J].\{v''''\} + [S].\{v''\} + [M].\{\ddot{v}\} = \{V_f{}'\} \qquad (17)$$

Based on the WallPanels Theory (WPT), the differential equation of the dynamic request of the massive column shown in Figure 6 is expressed, through equation (17), as:

$$-[J].\{q''''(x,t)\} + [S].\{q''(x,t)\} + [M].\{\ddot{q}(x,t)\} = \{V_f{}'(x,t)\} \qquad (18)$$

with: $[J]$ is the column stiffness matrix; $[S]$ is the lintel stiffness matrix (and for the massive pillar, modeled in this item of the thesis, it will be null); $[M]$ is the mass matrix of the abutment and $q(x,t)$is the function of the displacements dependent on space and time. Through the harmonic analysis of equation (18) and imposing the procedure for the separation of variables, it is written:

$$\frac{-j.u''''(x)}{u(x)} = \frac{-m.\ddot{g}(t)}{g(t)} = -\lambda_a{}^2 \qquad (19)$$

with: $q(x,t) = u(x).g(t)$. The characteristic equation of ODE written in space is expressed, via equation (19), as:

$$j.\omega^4 - \lambda_a{}^2 = 0 \qquad (20)$$

and by equation (20) solution is expressed:

$$\omega = \frac{\sqrt{\lambda_a}}{\sqrt{\sqrt{j}}} \qquad (21)$$

By calculating the stiffness, where j = E.I, together with Pfeil (p. 211) [16], concludes by the rigidity of the column shown in Figure 6, the following:

$$j = E.I = E.\beta.I_{z_{topo}} = 4{,}67\ x\ 10^4\ Pa\ .1{,}2793\ .\frac{10\ m\ .(5\ m)^3}{12} = 6{,}22326\ x\ 10^{12}\frac{N}{m} \qquad (22)$$

As presented inDziewolski[17]followed by the adjustment of the column stiffness using the coefficient$\alpha = {}^1/_{1,5}$ (for simple structures)and is expressed via equation:

$$\sqrt{\sqrt{j}} = \sqrt{\sqrt{\frac{j}{H^2}.\alpha}} = \sqrt{\sqrt{\frac{6{,}22326\ x\ 10^{12}\frac{N}{m}}{(100\ m)^2}.\frac{1}{1{,}5}}} = 105{,}29633 \qquad (23)$$

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Applying to equation (23) in equation (21) the first vibration frequencies of the bridge pillar, shown in Figure 6, are written as:

$$\omega = \frac{\sqrt{\lambda_a}}{105{,}29633} \tag{24}$$

V. COLUMN MODELING IN 5 FE APPLIED TO MODAL BRIDGE ANALYSIS

In order to exemplify the use of the linearly variable rectangular section stiffness matrix in bridge pillars (See Figure6), cross-sectional dimensions at the base $b_y = 12.5\ m$ and $b_z = 25\ m$and, at the top, $h_y = 5\ m$ and $h_z = 10\ m$, and the modes of vibration are obtained through modal analysis [18], modeling via five finite elements [19] and general formulation for *n* mass presented in Warburton [20] and matrix condensation [21]; and [22]. The material used in the bridge is reinforced concrete of resistance class C - 40 [23]. Therefore, the Longitudinal Elasticity Module will be $E = 35\ x 10^9 Pa$ and the Poisson Coefficient will be $\nu = 0{,}20$.In order to validate the example, modeling is performed using the ANSYS academic version software. It should be noted that for the vibration modes, the dimensions adopted for the cross section become irrelevant.

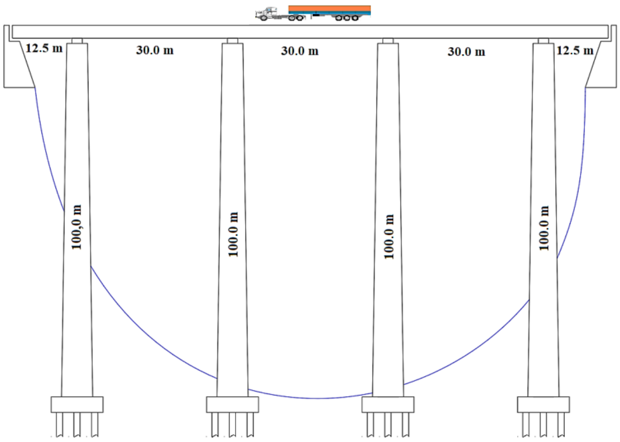

Fig. 6: Bridge with linearly variable section columns

From this analysis, the coefficients for the generation of the column stiffness matrix and vibration frequencies ω_i and autoversors λ_i^2, via the nullity of the determinant $|[K] - \lambda.[M]|$, are obtained in Table 1, as follows:

$$\lambda_1^2 = 6.16906\ x\ 10^5 \left(\frac{rad}{s}\right)^2;\ \lambda_2^2 = 7.14143\ x\ 10^5 \left(\frac{rad}{s}\right)^2;$$

$$\lambda_3^2 = 7.83961\ x\ 10^5 \left(\frac{rad}{s}\right)^2;\ \lambda_4^2 = 8.34502\ x\ 10^5 \left(\frac{rad}{s}\right)^2;$$

$$\lambda_5^2 = 8.71653\ x\ 10^5 \left(\frac{rad}{s}\right)^2 \tag{25.a - e}$$

$$\omega_1 = \sqrt{\frac{\lambda_1^2}{H^2}} \equiv 7.85434\ \frac{rad}{s} \equiv 1.25006\ Hz;$$

$$\omega_2 = 8.45070\ \frac{rad}{s};\quad \omega_3 = 8.85415\ \frac{rad}{s};$$

$$\omega_4 = 9.13511\ \frac{rad}{s};\quad \omega_5 = 9.33624\ \frac{rad}{s} \tag{26.a - e}$$

Table.1: Parameters for the generation of the stiffness matrix of the linearly variable section column

Bar Finite Element	1	2	3	4	5
$A\ (ADM)$	- 0.015	- 0015	- 0.015	- 0.015	- 0.015
$B\ (m)$	25.000	22.000	19.000	16.000	13.000
$C\ (ADM)$	- 0.075	- 0.075	- 0.075	- 0.075	- 0.075
$D\ (m)$	12.500	11.000	9.500	8.000	6.500
$\alpha_i \equiv \alpha_f\ (x\ 10^5)$	6.540	4.430	2.890	1.799	1.050
$k_i\ (x\ 10^{13})$ [N.m]	2.274	1.364	0.759	0.382	0.166
$k_f\ (x\ 10^{13})$ [N.m]	2.261	1.356	0.754	0.379	0.165
$a\ (x\ 10^{13})$ [N.m]	1.134	0.680	0.378	0.190	0.083

When considering the five finite elements, the Stiffness $[K]$ and Mass $[M]$ Matrices for the pillar are worth:

$$[K] = \begin{bmatrix} 1.701\,x\,10^{13} & 7.634\,x\,10^{11} & 3.178\,x\,10^{10} & 1.196\,x\,10^{9} & 3.891\,x\,10^{7} \\ 7.634\,x\,10^{11} & 1.019\,x\,10^{13} & 4.241\,x\,10^{11} & 1.596\,x\,10^{10} & 5.193\,x\,10^{8} \\ 3.178\,x\,10^{10} & 4.241\,x\,10^{11} & 5.666\,x\,10^{12} & 2.132\,x\,10^{11} & 6.939\,x\,10^{9} \\ 1.196\,x\,10^{9} & 1.596\,x\,10^{10} & 2.132\,x\,10^{11} & 2.848\,x\,10^{12} & 9.270\,x\,10^{10} \\ 3.891\,x\,10^{7} & 5.193\,x\,10^{8} & 6.939\,x\,10^{9} & 9.270\,x\,10^{10} & 1.238\,x\,10^{12} \end{bmatrix} \frac{N}{m}$$

$$[M] = \begin{bmatrix} 1.955\,x\,10^{7} & 9.995\,x\,10^{5} & 0 & 0 & 0 \\ 9.995\,x\,10^{5} & 1.227\,x\,10^{7} & 6.761\,x\,10^{5} & 0 & 0 \\ 0 & 6.761\,x\,10^{5} & 7.281\,x\,10^{6} & 4.374\,x\,10^{5} & 0 \\ 0 & 0 & 4.374\,x\,10^{5} & 4.030\,x\,10^{6} & 2.685\,x\,10^{5} \\ 0 & 0 & 0 & 2.685\,x\,10^{5} & 1.984\,x\,10^{6} \end{bmatrix} kg$$

a) Validation via ANSYS

Figure 7 shows a group of vibration modes, for the massive column of the bridge shown in Figure 6, modeling in the ANSYS software. 62,468 nodes and 13,635 finite elements were used, producing a mesh with 93.99%.

The validation of the first vibration frequency f_1 (using equation 26 a) is processed and results in:

$$f_1 = \frac{\omega_1}{2\pi} = \frac{7.85434\ \frac{rad}{s}}{2\pi} = 1.25005\ Hz \tag{27}$$

Comparing the first vibration frequency by modeling in ANSYS, see Figure 7, with the value presented in equation (27), an approximation of:

$$\Delta(\%) = \frac{(1.25005 - 1.23300)\ Hz}{1.25005\ Hz}.100\% = 1.36\ \% \tag{28}$$

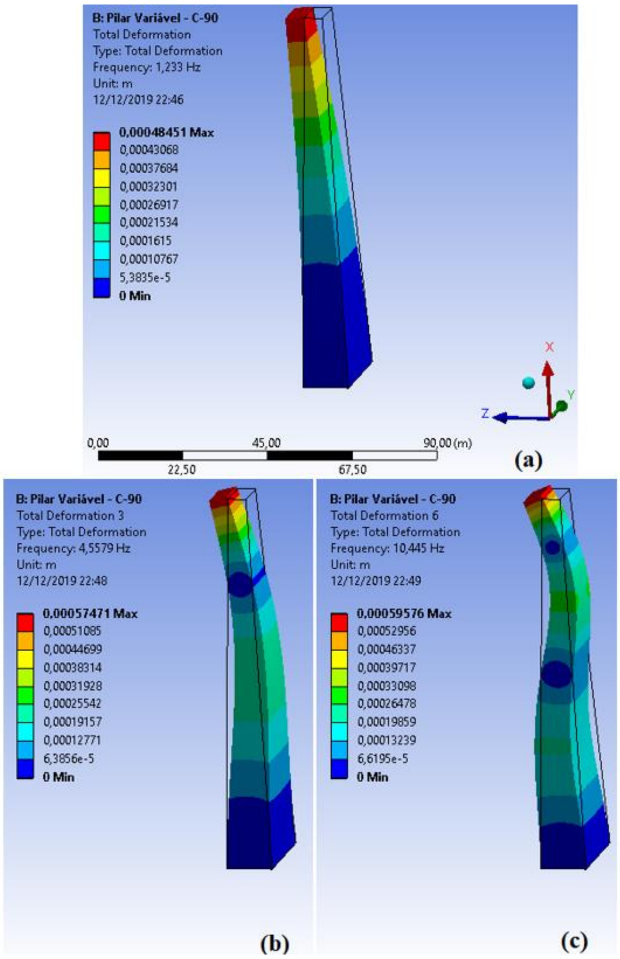

Fig. 7: Vibration modes in bridges with linearly variable section columns: (a) 1st mode, (b) 2nd mode, (c) 3rd mode

The 1.36% discrepancy between the first vibration frequency, via manual calculation by finite bar elements and by modeling in the ANSYS software, is due to the small number of bar elements used in manual modeling. However, the formulation present here is quite satisfactory to verify the order of magnitude of the results obtained via modeling in commercial software.

b) Validation via CMT

Using the first root of the polynomial presented in equation (25a), the first vibration frequency of the abutment (via CMT) is expressed as:

$$\omega_1 = \frac{\sqrt{6.16906\ x\ 10^5}\ rad/s}{105.29633} = 7.45927\ \frac{rad}{s} \tag{29 a}$$

and in fundamental frequency, there is:

$$f_1 = \frac{\omega_1}{2\pi} = 1.18718\ Hz \tag{29 b}$$

concluding by divergence in relation to the modeling performed in ANSYS, see Figure 7, the following:

$$\Delta(\%) = \frac{|f_{TMC} - f_{ANSYS}|}{f_{TMC}}.100\%$$

$$= \frac{|1.18718 - 1.233|\ Hz}{1.18718\ Hz}.100\% = 3.87\ \% \tag{29 c}$$

It is observed that the percentage difference by the CMT was greater than by the bar FEM, this due to the interpolation performed in the coefficientβ of equation (22). As well as, it is verified the use of the adjustment coefficient α in equation (23). While in equation (28) the analysis is processed by the finite element method, eliminating the imposition of such a coefficientα. Even so, it is possible to satisfactorily validate the modal analysis of the column shown in Figure 6, both by FEM and CMT.

VI. CONCLUSION

In this article, the terms of the Flexibility Matrix for a variable section were obtained through the internal energy activated by the internal forces acting on the beam element for the Derivative Systems. The contribution of this publication is the obtention of the stiffness matrix by parameters easily obtained from the flexibility matrix which takes into account the variation of the dimensions of the element cross section. When analyzing the shape factor k_c, it was found to be a constant value according to the section shape. After determining the terms of the Flexibility Matrix, the terms of the stiffness matrix are obtained as shown in this work.

Finally, it is made explicit that the expressions presented here for α_i, α_f, ε and β_f are valid only for cases of variable cross-section along the longitudinal axis of the pillar. In case of a constant section, one should resort to the expressions widely postulated in technical texts, under penalty of mathematical indetermination by these explicit expressions for $\alpha, \beta, \varepsilon$ and β_f.

This contribution emphasizes the achievement of the exact stiffness matrix for such a cross-section configuration, and the application of such a matrix is performed in the calculation of the vibration frequencies of the pillars of a bridge with a tray positioned at 100 meters, this relative to the base, of the mentioned columns.

For the first vibration frequency, with the column being subdivided into five finite bar elements, an approximation of 1.36% is concluded. Such an approximation is excellent, due to the number of finite elements used in the discretization of the 100meters high column, the validation resulting from modeling in ANSYS.

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